Phys 406 Practice Final Exam

2nd May 2006

1. (25 pts) In the Friedmann-Robertson-Walker background metric,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

we want to compute the aging process of a geodesically moving (freely moving without external nongravitational forces) particle m using the conserved momenta method.

a) (5 pts) Write down the conserved momenta and call them k_x, k_y, k_z respectively.

b) (10 pts) Find an equation for $(\frac{dt}{d\tau})^2$ where τ is the proper time.

c) (10 pts) How much does the particle age (according to the usual clock of the rest frame of the particle) between cosmological time $t = t_0$ and time $t = t_1 > t_0$. Express your answer as an integral.

answer

a) The metric does not depend on $\vec{x}$. Hence,

$$\frac{dx_1}{d\tau} = k_x$$
$$\frac{dx_2}{d\tau} = k_y$$
$$\frac{dx_3}{d\tau} = k_z$$

b) The square norm of the momentum is

$$p^2 = -m^2$$

giving

$$-m^2(\frac{dt}{d\tau})^2 + (k_x^2 + k_y^2 + k_z^2)\frac{m^2}{a^2} = -m^2$$
$$(\frac{dt}{d\tau})^2 = 1 + (k_x^2 + k_y^2 + k_z^2)\frac{1}{a^2}$$

c) Integrating the result of part b), we see

$$\tau = \int d\tau = \int_{t_1}^{t_2} \frac{dt}{\sqrt{1 + (k_x^2 + k_y^2 + k_z^2)\frac{1}{a^2(t)}}}$$

2. (25 pts) In the Schwarzschild background,

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega^2$$

we have the following conserved momenta for a photon:

$$p^0 = \frac{\tilde{E}}{1 - \frac{2M}{r}}$$

$$p^\phi = \frac{\tilde{L}}{r^2}$$

a) (10 pts) Find the radial equation of motion (differential equation in terms of proper time) for the photon in the plane $\theta = \pi/2$.

b) (15 pts) Find a stable orbit if there is one. Otherwise, justify the lack of a stable orbit.

answer

a) The scalar norm of the 4-momentum vector is

$$\begin{aligned} p^\mu p_\mu &= g_{00}(p^0)^2 + g_{11}(p^r)^2 + g_{22}(p^\theta)^2 + g_{33}(p^\phi)^2 \\ &= \frac{-1}{1 - \frac{2M}{r}}\tilde{E}^2 + \frac{1}{1 - \frac{2M}{r}}(\frac{dr}{d\tau})^2 + \frac{\tilde{L}^2}{r^2} \\ &= 0 \end{aligned}$$

Therefore, we find

$$(\frac{dr}{d\tau})^2 = \tilde{E}^2 - (1 - \frac{2M}{r})\frac{\tilde{L}^2}{r^2} \qquad (1)$$

b) The stable radius lies at the location where the "potential" from Eq. (1)

$$V^2 = (1 - \frac{2M}{r})\frac{\tilde{L}^2}{r^2}$$

has a minimum where

$$l \equiv m\tilde{L}.$$

Hence, we set

$$\frac{d}{dr}V^2 = \frac{2\tilde{L}^2 M}{r^4} - \frac{2\tilde{L}^2}{r^3}(1 - \frac{2M}{r}) = 0$$

giving

$$\tilde{L}^2(3M - r) = 0.$$

Hence, we find

$$r = 3M.$$

Unfortunately, this is not a stable orbit since

$$\frac{d^2}{dr^2}V^2 = \frac{6\tilde{L}^2(r-4M)}{r^5}|_{r=3M} < 0$$

Hence, the potential is concave down and unstable.

3. (25 pts) Outside of a spherically symmetric star of mass M, the radial motion of a planet is described by the equation

$$(\frac{dr}{d\tau})^2 = \tilde{E}^2 - (1-\frac{2M}{r})(\frac{\tilde{L}^2}{r^2}+1)$$

in the usual notation of

$$\tilde{L} \equiv \frac{p_\phi}{m}$$

and

$$\tilde{E} \equiv \frac{-p_0}{m}.$$

a) (10 pt) Find the differential equation for $r(\phi)$ (radius as a function of azimuthal angle) in the plane $\theta = \pi/2$. [The Schwarzschild metric is defined as

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{(1-\frac{2M}{r})} + r^2 d\Omega^2$$

]

b) (10 pt) Solve the differential equation in the case $M = 0$ for the function $r(\phi)$ (the answer will contain $\tilde{E}^2$and $\tilde{L}^2$. You may find the following integral useful.

$$\int \frac{dr}{r\sqrt{c_1 r^2 - 1}} = -ArcTan[\frac{1}{\sqrt{c_1 r^2 - 1}}] + constant$$

c) (5 pt) Sketch the answer to b). Clearly mark the closest approach to the origin. (i.e. Give the value of the closest approach distance.)

answer

a) Elementary identities give

$$\frac{dr}{d\phi} = \frac{dr}{d\tau}/\frac{d\phi}{d\tau}$$

Since

$$\tilde{L} = g_{\phi\phi}\frac{p^\phi}{m} = r^2 \sin^2\theta \frac{d\phi}{d\tau}.$$

Hence,

$$\frac{dr}{d\phi} = \pm\frac{r^2}{\tilde{L}}\sqrt{\tilde{E}^2 - (1-\frac{2M}{r})(\frac{\tilde{L}^2}{r^2}+1)}.$$

b) For $M = 0$, we find

$$\begin{aligned}\frac{dr}{d\phi} &= \pm\frac{r^2}{\tilde{L}}\sqrt{\tilde{E}^2 - (\frac{\tilde{L}^2}{r^2} + 1)} \\ &= \pm\frac{r}{\tilde{L}}\sqrt{(\tilde{E}^2 - 1)r^2 - \tilde{L}^2} \\ &= \pm r\sqrt{c_1 r^2 - 1}\end{aligned}$$

where

$$c_1 \equiv \frac{(\tilde{E}^2 - 1)}{\tilde{L}^2}$$

Thus, using the given identity, we have

$$-ArcTan[\frac{1}{\sqrt{c_1 r^2 - 1}}] = \pm\phi - c$$

where c is an arbitrary integration constant. Thus,

$$\tan^2(c \mp \phi) = \frac{1}{c_1 r^2 - 1}$$

or

$$c_1 r^2 = 1 + \cot^2(c \mp \phi) = \csc^2(c \mp \phi)$$

Hence,

$$r = \frac{1}{\sqrt{c_1}}\csc(c \mp \phi)$$

or

$$r\sin(c \mp \phi) = \frac{1}{\sqrt{c_1}} = \frac{\tilde{L}}{\sqrt{\tilde{E}^2 - 1}}$$

c) A straight line with the closest approach to origin of $\frac{\tilde{L}}{\sqrt{\tilde{E}^2-1}}$.

4. (20 pts) Suppose you are given that the 4-velocity of an observer is

$$U^\mu = (\sqrt{1 + k_x^2/a^2}, U^1, 0, 0)$$

and there is a photon with momenta

$$p^\mu = (E/a, 0, p^2, 0, 0)$$

in the background metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

a) (10 pts) Find the components U^1 and p^2 in terms of $\{k_x, E, a\}$.

b) (10 pts) What is the energy of the photon measured by the observer's rest frame? (Express the answer in terms of k_x, E, and a.)

answer

a) By the normalization condition

$$U^\mu U_\mu = -1,$$

we see

$$U^1 = \pm \frac{k_x}{a^2}.$$

Similarly, for the photon,

$$p^2 = \pm \frac{E}{a^2}$$

b) The energy of the photon measured by the observer is

$$\bar{E} = -U^\mu p^\nu g_{\mu\nu} \tag{2}$$

Hence,

$$\bar{E} = \frac{E}{a}\sqrt{1 + k_x^2/a^2}$$

5. (25 pts) Consider the FRW metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

a) (15 pts) Compute the Christoffel symbol $\Gamma^1_{\alpha\beta}$.

b) (10 pts) Suppose a particle is freely along a geodesic with nonvanishing U^1. Find a first order ordinary differential equation for U^1 in terms of $\frac{\dot{a}}{a}$ assuming $U^0(\tau)$ and $t(\tau)$ is known.

answer

a) Using the equation

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda})$$

we find

$$\begin{aligned}
\Gamma^1_{\alpha\beta} &= \frac{1}{2} g^{11}(g_{\alpha 1,\beta} + g_{\beta 1,\alpha} - g_{\alpha\beta,1}) \\
&= \frac{1}{2} g^{11} g_{11,0}(\delta_{\alpha 1}\delta_{\beta 0} + \delta_{\alpha 0}\delta_{\beta 1})
\end{aligned}$$

Hence the only nonvanishing components are for $i, j \in \{1, 2, 3\}$

$$\Gamma^1_{\alpha\beta} = \frac{\dot{a}}{a}(\delta_{\alpha 1}\delta_{\beta 0} + \delta_{\alpha 0}\delta_{\beta 1})$$

b) The geodesic equation

$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Using the results of part a), we find

$$\frac{d}{d\tau}U^1 + 2\frac{\dot{a}}{a}U^0U^1 = 0$$

6. (30 pts) For the FRW metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2,$$

suppose you are given that the nonvanishing Ricci tensor components are

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{ii} = 2\dot{a}^2 + a\ddot{a} \text{ for } i \in \{1,2,3\}.$$

a) (15 pts) Write down the $0-0$ and $1-1$ components of Einstein equations in terms of a perfect fluid stress energy tensor (with the fluid at rest in the FRW coordinates – i.e. rest in the comoving frame). Denote the fluid energy density and pressure in the comoving frame as $\rho(t)$ and $P(t)$.

b) (15 pts) Given that

$$\Gamma^0_{00} = 0$$

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a}$$

$$\Gamma^\nu_{\nu 0} = 3\frac{\dot{a}}{a}$$

write

$$T^{0\nu}{}_{;\nu}$$

in terms of $\frac{\dot{a}}{a}$, ρ, and P.

answer

a) In general, Einstein equations are given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}.$$

Since we need R, we first compute

$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + 3[2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}] = 6[\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}].$$

Hence,

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= -3\frac{\ddot{a}}{a} + \frac{6}{2}[\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}] \\ &= 3\frac{\dot{a}^2}{a^2} = 8\pi T_{00} \end{aligned}$$

$$\begin{aligned} R_{11} - \frac{1}{2}g_{11}R &= 2\dot{a}^2 + a\ddot{a} - \frac{6}{2}a^2[\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}] \\ &= -\dot{a}^2 - 2\ddot{a}a = 8\pi T_{11} \end{aligned}$$

Since the perfect fluid stress tensor is given as

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + Pg_{\mu\nu},$$

and is at rest, we find

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi}{3}\rho$$

and

$$-\dot{a}^2 - 2\ddot{a}a = 8\pi P a^2$$

b) $T^{0\nu}{}_{;\nu} = 0$ yields

$$T^{0\nu}{}_{,\nu} + \Gamma^0_{\nu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\nu\lambda}T^{0\lambda} = 0.$$

This yields

$$T^{00}{}_{,0} + 3\Gamma^0_{11}T^{11} + \Gamma^\nu_{\nu 0}T^{00} = 0$$

where we have made use of noting the nonvanishing components of Γ. Since $T^{00} = \rho$ and $T^{11} = \frac{1}{a^2}P$, we see

$$\dot{\rho} + 3\frac{\dot{a}}{a}P + 3\frac{\dot{a}}{a}\rho = 0.$$

7. (25 points) Describe in words the derivation of Einstein's equations. Try to use the following words: 1) energy density 2) Equivalence principle 3) two derivatives 4) tensor 5) stress energy tensor 6) Riemann tensor 7) Bianchi identities 8) weak field limit. Using 5 of the 8 words to meaningfully account for the logic of derivation of the Einstein equations will be result in the awarding of the full 25 points.

answer

Any 5 of the following points will be awarded a full score.

1. Noting that Newton's gravitational law must be incorporated into relativity, the energy density must be a source term for the Newtonian potential.

2. Using the Equivalence principle, one deduces that the gravitational potential can be identified with a component of a metric.

3. Because the Poisson's equation for the gravitational potential involves two derivatives, the gravitational equation must involve at least two derivatives of the metric.
4. Because of the Equivalence principle, the equation must be in a tensorial form.
5. A tensor that contains the energy density as a component is naturally identified as the stress energy tensor.
6. The simplest nontrivial tensors that can be made using two derivatives of the metric is the Riemann tensor and all its contractions.
7. Bianchi identities, required by the conservation of the stress energy tensor, serve to fix the relative coefficient between the Ricci tensor and the Ricci scalar parts of the Einstein equations.
8. Weak field limit fixes the coefficient in front of the stress energy tensor in the Einstein equations.