Physics 406 Exam 1

1. (40 pts) Suppose the separation between event A and event B is given by

$$\Delta x^\alpha = (1, 0, 0, -1)$$

a) (10 pt) Compute what Δx^α looks like when boosted with speed v along the z-axis.

b) (10 pt) Is Δx^α spacelike, null, or timelike?

c) (10 pt) Compute what Δx_α looks like under the same boost as in a).

d) (10 pt) How does $\Delta x^\alpha \Delta x_\alpha$ transform under Lorentz transformations?

answer

a) Applying Lorentz transformations, we have

$$\Delta x'^\alpha = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \gamma(1+v) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

b) Since $\Delta x^\alpha \Delta x_\alpha = 0$, the interval is null.

c) Since we have $\Delta x_\alpha = \eta_{\alpha\beta} \Delta x^\beta$, we find

$$\Delta x'_\alpha = -\gamma(1+v) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

d) By construction $\Delta x^\alpha \Delta x_\alpha$ is invariant under Lorentz transformations.

2. (20 pt) A particle with mass m is moving with relativistic energy E_1 in the lab frame collides head on with another particle with identical mass m initially *at rest* in the lab frame. Suppose after the collision, there are four particles with mass m. What is the minimum E_1 for which this collision reaction can occur? (i.e. What is the minimum E_1 for which there can be four particles after the collision?)

answer

Conservation of 4-momentum yields

$$p_1 + p_2 = p_3 + p_4 + p_5 + p_6$$

where on the right hand side we have 4 particles. If we take the relativistically invariant square of both sides, we have

$$(p_1 + p_2)^2 = (p_3 + p_4 + p_5 + p_6)^2. \tag{1}$$

The left hand side of Eq. (1) simplifies to

$$-(p_1^2 + p_2^2 + 2p_1 \cdot p_2) = 2m^2 + 2mE_1$$

in the lab frame. Since we want the minimum energy, it is convenient to evaluate the right hand side of Eq. (1) in the center of momentum frame, in which each of the 4 particles will be at rest after the collision:

$$(p_3 + p_4 + p_5 + p_6)^2 = -(E_3 + E_4 + E_5 + E_6)^2 = -(4m)^2.$$

Hence, we find

$$E_1 = 7m.$$

3. (20 pt) You have seen in class that the differential operator in the equation

$$[\partial_t^2 - \partial_x^2]\phi(t,x) = 0.$$

is Lorentz invariant under the transformation $(t,x) \rightarrow (\gamma t + \gamma v x, \gamma x + \gamma v t)$. Is the differential operator in the following equation Lorentz invariant?

$$[\partial_t^4 - 2\partial_t^2\partial_x^2 + \partial_x^4]\phi(t,x) = 0$$

Why or why not?

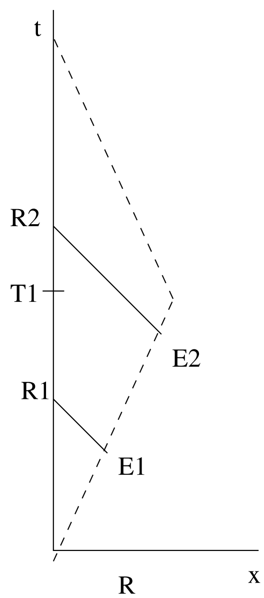
answer

Yes, the operator is Lorentz invariant since

$$(\partial_t^2 - \partial_x^2)^2 = \partial_t^4 - 2\partial_t^2\partial_x^2 + \partial_x^4$$

and you already know $(\partial_t^2 - \partial_x^2)$ is Lorentz invariant.

4. (20 pts) Suppose a light bulb which flashes in its rest frame every time interval τ' is receding away from you at a constant velocity v for a time period T_1 and then moves towards you at constant velocity $-v$ for a time period T_1 . Assume that the light bulb passes your position (origin) at time $t = 0$. The following spacetime diagram shows the worldline of the lightbulb and the worldline of two consecutive light pulses during the initially outgoing time period T_1.



a) Let the earlier of the two emission events ($E1$) take place at time t_1. Let the time between the emission events in your frame be labeled τ. Label the two emission events ($E1$ and $E2$) and the two reception events ($R1$ and $R2$) of the light pulses with appropriate spacetime coordinates.

answer

We can label the (t,x) coordinates of the events as follows:

$$E1: \ (t_1, vt_1)$$

$$E2: \ (t_1 + \tau, v(t_1 + \tau))$$

$$R1: \ (t_1 + vt_1, 0)$$

$$R2: \ (t_1 + \tau + v(t_1 + \tau), 0)$$

b) Compute the time interval between the two reception events in terms of τ'.

answer

The time interval of reception events is simply

$$\tau(1+v)$$

from $R1$ and $R2$ of part a). To relate τ to τ', we must compute what $E1$ and $E2$ looks like in the bulb's rest frame.

$$E1: \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t_1 \\ vt_1 \end{pmatrix} = \begin{pmatrix} \gamma t_1 - \gamma v^2 t_1 \\ 0 \end{pmatrix} = \begin{pmatrix} t_1/\gamma \\ 0 \end{pmatrix}$$

$$E2: \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t_1+\tau \\ v(t_1+\tau) \end{pmatrix} = \begin{pmatrix} \gamma(t_1+\tau) - \gamma v^2(t_1+\tau) \\ 0 \end{pmatrix} = \begin{pmatrix} (t_1+\tau)/\gamma \\ 0 \end{pmatrix}$$

Hence, we have

$$\tau' = \frac{\tau}{\gamma}$$

which we would have guessed from Lorentz transformations. Hence, the time interval between the two reception events is

$$\gamma\tau'(1+v) = \tau'\sqrt{\frac{1+v}{1-v}}.$$