Physics 406 Exam 2

Thursday, 4-6-06, 1:20-2:10 PM

1. (25 pts) Suppose a point-like test mass (of mass q) is released from rest a distance d away from approximately two dimensional membrane of mass density σ and infinite extent (i.e. the amount of mass contained the membrane in area a is σa). Assume that the gravitational field generated by the membrane is nonrelativistic and that the membrane is oriented parallel to the xy plane and is located at $z = 0$.

a) (5 pt) Write down the acceleration of the test mass as a function of z (the distance to the membrane) using Newton's Law. (Hint: Poisson's equation for gravitation is $\nabla^2\phi = 4\pi\rho$ where ρ is the mass density)

---

answer

Poisson's law gives

$$\nabla^2\phi = 4\pi\sigma\delta(z).$$

Taking a pill box of cross sectional area a, we have in analogy with electrostatics

$$|\vec{\nabla}\phi|2a = 4\pi\sigma a$$

Hence, we find

$$\frac{d^2 z}{dt^2} = 2\pi\sigma$$

---

b) (10 pt) Write down the geodesic equation for the test mass position $x^\mu(\tau)$ where τ is the proper time (leave the answer in terms of ${\Gamma^\mu}_{\alpha\beta}$).

---

answer

The geodesic equation is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

---

c) (10 pt) Assuming that $\frac{dx^0}{d\tau} = 1$ and non-relativistic speed approximation, compute the Christoffel symbol component Γ^z_{tt} (i.e. Γ^3_{00} in the (t, x, y, z) coordinates).

---

answer

The geodesic equation in the nonrelativistic limit is

$$\frac{d^2 z}{dt^2} + \Gamma^3_{00} = 0.$$

Hence, we find

$$\Gamma^3_{00} = -2\pi\sigma$$

---

2. (25 pts) Suppose you are given that at point P in the coordinate system $x^\mu = (t, x, y, z)$ that

$$\frac{\partial V^\lambda}{\partial x^\mu}|_P = 0$$

and

$$V^\lambda|_P = (1, 0, 0, 0)$$

where V^λ is a covariant vector under general coordinate transformations. Compute

$$\frac{\partial \bar{V}^0}{\partial \bar{x}^0}|_P$$

(at point P) in the coordinate system $\bar{x}^\mu = (u, v, y, z)$ where

$$u = t - x$$

$$v = t + x$$

assuming that the point P is located at $(t, x, y, z) = (1, 0, 0, 0)$. Hint: Either remember the inhomogeneous transformation rule of $\partial_\mu V^\lambda$ or even more simply start from the definition of how V^λ transforms under coordinate transformation.

---

answer

Simply expand:

$$\begin{aligned}
\frac{\partial \bar{V}^0}{\partial \bar{x}^0}|_P &= \frac{\partial}{\partial \bar{x}^0}[\frac{\partial \bar{x}^0}{\partial x^\lambda} V^\lambda]_P \\
&= \frac{\partial x^\gamma}{\partial \bar{x}^0}\frac{\partial^2 \bar{x}^0}{\partial x^\gamma \partial x^\lambda} V^\lambda|_P + \frac{\partial \bar{x}^0}{\partial x^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^0}\frac{\partial V^\lambda}{\partial x^\gamma}|_P \\
&= \frac{\partial x^\gamma}{\partial \bar{x}^0}\frac{\partial}{\partial x^\gamma}(1) = 0
\end{aligned}$$

---

3. (30 pts) Suppose on a two dimensional surface parameterized by (x, y) (valid for $x \neq 0$) with a metric

$$ds^2 = Q^2 dx^2 + x^4 dy^2,$$

there is a curve C parameterized by $(x = x_0, y)$ where y runs from 0 to 1. (Here, Q is a constant.) If there is a vector $\vec{V}$ on the curve at position $(x = x_0, y = 0)$ with magnitude $\vec{V}(x = x_0, y = 0) = \hat{y}$ ($\hat{y}$ is the usual basis unit vector), what is the value of the vector when it is parallel transported along the curve C to $(x = x_0, y = 1)$? (Hint: $\Gamma^2_{2\beta} = \delta_{\beta 1}\frac{2}{x}$ where the notation is as usual, i.e. $x^1 = x$ and $x^2 = y$. Hence, only $\Gamma^1_{2\beta}$ needs to be computed.)

---

answer

We start with the geodesic equation

$$\frac{dx^\gamma}{d\lambda}\nabla_\gamma V^\alpha = 0.$$

Since we have parameterized the curve in terms of y, we have

$$\partial_2 V^\alpha + \Gamma^\alpha_{2\lambda} V^\lambda = 0.$$

Writing this out, we have

$$\partial_2 V^1 + \Gamma^1_{2\lambda} V^\lambda = 0$$

$$\partial_2 V^2 + \Gamma^2_{2\lambda} V^\lambda = 0$$

Computing

$$\begin{aligned}\Gamma^1_{2\beta} &= \frac{1}{2} g^{1\lambda}(g_{2\lambda,\beta} + g_{\beta\lambda,2} - g_{2\beta,\lambda}) \\ &= \frac{1}{2} g^{11}(g_{21,\beta} + g_{\beta 1,2} - g_{2\beta,1}) \\ &= \frac{-2x^3}{Q^2}\delta_{\beta 2}.\end{aligned}$$

Hence, our equations turn into

$$\partial_2 V^1 - \frac{2x_0^3}{Q^2} V^2 = 0$$

$$\partial_2 V^2 + \frac{2}{x_0} V^1 = 0$$

Taking the derivative of the first with respect to ∂_2, we find

$$\partial_2^2 V^1 - \frac{2x_0^3}{Q^2}\partial_2 V^2 = \partial_2^2 V^1 - \frac{2x_0^3}{Q^2}(-\frac{2}{x_0}V^1) = \partial_2^2 V^1 + \frac{4x_0^2}{Q^2}V^1 = 0.$$

Since V^λ only depends on y along the curve of interest, we find

$$V^1 = A\cos(\frac{2x_0}{Q}y) + B\sin(\frac{2x_0}{Q}y).$$

Since $V^1 = 0$ at $y = 0$, we have

$$V^1 = B\sin(\frac{2x_0}{Q}y).$$

Since $V^2 = 1$ at $y = 0$, we have the second boundary condition giving

$$V^2 = \frac{Q^2}{2x_0^3}\partial_2 V^1 = \frac{Q}{x_0^2}B\cos(\frac{2x_0}{Q}y)|_{y=0} = 1.$$

Hence,

$$V^1 = \frac{x_0^2}{Q}\sin(\frac{2x_0}{Q}y)$$

$$V^2 = \cos(\frac{2x_0}{Q}y)$$

---

4. (20 pts) Using the doubly contracted Bianchi identity

$$\nabla_\alpha(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R) = 0,$$

find the partial differential equation satisfied by the function $K(x)$

$$R_{\alpha\gamma} = K(x)g_{\alpha\gamma}.$$

---

answer

$$R_{\alpha\gamma} = K(x) g_{\alpha\gamma}$$

$$R = 4K(x)$$

Since the covariant derivative of the metric vanishes, we have

$$\nabla_\alpha [R^{\alpha\gamma} - \frac{1}{2} g^{\alpha\gamma} R] = 0$$

implying

$$K_{,\alpha} g^{\alpha\gamma} - 2 g^{\alpha\gamma} K_{,\alpha} = -g^{\alpha\gamma} K_{,\alpha} = 0$$

Hence, K must be a constant.
