PHYSICS 406: PROBLEM SET 7

Due: Tuesday, May 2

1.: Show that in radiation dominated universe,

$$R^{ab}R_{ab} \to \infty$$

as $a \to 0$. Hence, even for an equation of state of 1/3, there is a big bang singularity.

answer:

Consider the trace of Einstein tensor:

$$G^a{}_a = R - 2R = -R = 8\pi T$$

where $T \equiv T^a{}_a$. Hence, we can rewrite Einstein's equations as

$$\begin{aligned} R_{ab} &= 8\pi T_{ab} + \frac{1}{2}g_{ab}R \\ &= 8\pi T_{ab} - \frac{1}{2}g_{ab}8\pi T \\ &= 8\pi(T_{ab} - \frac{1}{2}g_{ab}T). \end{aligned}$$

For a perfect fluid, the stress energy tensor is

$$T_{ab} = (\rho + P)U_aU_b + Pg_{ab}$$

while its trace is

$$T = -(\rho + P) + 4P = 3P - \rho.$$

As we learned in class, since for equation of state of $P/\rho = 1/3$, we have $T = 0$, we can neglect the T term in R_{ab} and write

$$\begin{aligned} R_{ab}R^{ab} &= 64\pi^2(T_{ab}T^{ab}) \\ &= 64\pi^2[(\rho + P)U_aU_b + Pg_{ab}][(\rho + P)U^aU^b + Pg^{ab}] \\ &= 64\pi^2[(\rho + P)(\rho + P)U^aU^bU_aU_b + Pg_{ab}(\rho + P)U^aU^b + \\ &\quad +(\rho + P)U_aU_bPg^{ab} + Pg_{ab}Pg^{ab}] \\ &= 64\pi^2[(\rho + P)^2 - 2P(\rho + P) + 4P^2] \\ &= 64\pi^2[\rho^2 + 3P^2] \\ &= 64\pi^2\rho^2[\frac{4}{3}] = \frac{256\pi^2}{3}\rho^2. \end{aligned}$$

We derived in class that

$$\rho = \rho_0(\frac{a_0}{a})^4.$$

Hence, for $a \to 0$, $\rho \to \infty$ and $R_{ab}R^{ab} \to \infty$.

2.: Because electrons combined with protons to form neutral hydrogen at temperature of about 1 eV, majority of the cosmic background photons that we see today last scattered off of charged particles at that temperature from a spherical surface very far away. This is called the "last scattering surface."

a): Given that the temperature of the cosmic background photon today is about 10^{-3}eV, compute the redshift z_* at the last scattering surface.

answer:

As we have shown in class, redshift of light due to cosmological expansion is given as

$$(1 + z(t)) = \frac{a(t_0)}{a(t)}$$

and the entropy conservation leads to

$$T \propto \frac{1}{a}.$$

Hence, we have

$$(1 + z) = \frac{T}{T_0}.$$

Plugging in $T \sim 1$ eV and $T_0 \sim 10^{-3}$eV, we find the redshift to be

$$z_* \sim 10^3$$

at the last scattering surface.

(b): Compute the physical distance (on the homogeneous spacelike hypersurface today) from Earth to the last scattering surface. (i.e. the last scattering surface occupies a fixed coordinate distance with respect to us today. Compute the physical distance between that surface and us where the spatial metric is defined on the homogeneous spacelike hypersurface today. Express your answer in cm units to order of magnitude accuracy.)

answer:

The physical distance is given as

$$l = a(t_0) \int dr$$

For most of the late time history of the universe, one should assume a constant equation of state $w \equiv P/\rho$ model with $K = w = 0$ (K is the spatial curvature parameter in FRW metric). This led to the equation

$$t = \frac{2}{3H_0(1+z)^{3/2}}.$$

This yields

$$dt = \frac{-1}{H_0} \frac{dz}{(1+z)^{5/2}}.$$

Since the photons travel on null geodesics, we have $ds^2 = 0$ yielding

$$\begin{aligned} \int dr &= \int_{t_l}^{t_0} \frac{dt}{a} = \frac{-1}{H(t_0)a(t_0)} \int_{z_l}^{0} \frac{dz}{(1+z)^{3/2}} \\ &= \frac{2}{H(t_0)a(t_0)}[1 - \frac{1}{(1+z_l)^{1/2}}] \end{aligned}$$

Substituting $z_l = 10^3$, we find

$$\int dr \approx \frac{2}{H(t_0)a(t_0)}(0.97).$$

Hence today's physical distance to the last scattering surface is

$$l = \frac{1.94}{H(t_0)} \approx \frac{1.94}{65\text{km/s/Mpc}}(3 \times 10^5 \text{km/s}) \approx 9 \times 10^3 \text{ Mpc} \approx 3 \times 10^{28} \text{cm}.$$

That is a pretty large distance! The modern CMB measurements are seeing that far away!

3.: Problem 9.3

answer:

We have shown in class that the luminosity distance is given by

$$d_L = r a(t_0)[1+z] \tag{0.1}$$

where r is the coordinate location of the object emitting the light. We can compute r for a cosmology containing just a cosmological constant and the matter in a flat universe ($\Omega_0 = 1$). As we have done in the lectures, start with the null geodesic condition

$$ds^2 = 0 = -dt^2 + a^2(t) dr^2.$$

Integrating, we find

$$\begin{aligned} -\int dr &= \int \frac{dt}{a(t)} \\ &= \int da \frac{1}{\dot{a}a} \\ &= \int da \frac{1}{Ha^2} \qquad (0.2) \end{aligned}$$

where we have changed integration variables from t to a and placed ourselves at the origin (hence $dr/dt < 0$). Using the Friedmann equation (G_{00} component of Einstein equations), we find

$$\begin{aligned} H^2 &= \frac{8\pi}{3}(\rho_M + \rho_\Lambda) \\ &= \frac{8\pi}{3}(\rho_M(t_0)(\frac{a(t_0)}{a})^3 + \rho_\Lambda) \\ &= \frac{8\pi}{3}(\rho_M(t_0)(1+z)^3 + \rho_\Lambda) \\ &= \frac{8\pi}{3}\frac{3H_0^2}{8\pi}(\Omega_0(1+z)^3 + \Omega_\Lambda) \end{aligned}$$

where

$$a_0/a \equiv 1 + z.$$

Inserting this into Eq. (0.2), we find

$$\begin{aligned} -\int_r^0 dr' &= \int_a^{a_0} da' \frac{1}{a'^2 H_0 \sqrt{\Omega_0 (1+z')^3 + \Omega_\Lambda}} \\ r &= \frac{1}{a_0} \int_0^z dz' \frac{1}{H_0 \sqrt{\Omega_0 (1+z')^3 + \Omega_\Lambda}} \end{aligned}$$

Putting this into Eq. (0.1), we arrive at the luminosity distance:

$$d_L = [1+z] \int_0^z dz' \frac{1}{H_0 \sqrt{(\Omega_0 (1+z')^3 + \Omega_\Lambda)}}$$

4.: Problem 9.4

answer:

see solutions in the back of the book.