Lec 10: Covariant Derivative

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1 How do we make a tensor using a derivative?

In special relativity, to make larger rank (number of indices) tensors from lower rank tensors, one can take derivatives of lower rank tensors. the reason is that the operator ∂_μ obeys the tensor transformation rule under Lorentz transformations no matter what it is operating on. One then naturally asks whether ∂_μ obeys the tensor transformation rules under general coordinate transformations. The answer is no, as we will now show.

Suppose you take a derivative of a vector V^ν as

$$\partial_\mu V^\nu$$

and ask how it transforms under general coordinate transformation:

$$\begin{aligned}\partial_\mu V^\nu \rightarrow \frac{\partial}{\partial x'^\mu} V'^\nu &= \frac{\partial}{\partial x'^\mu}[\frac{\partial x'^\nu}{\partial x^\lambda} V^\lambda] \\ &= \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\gamma \partial x^\lambda} V^\lambda + \frac{\partial x'^\nu}{\partial x^\lambda}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial V^\lambda}{\partial x^\alpha}. \end{aligned} \tag{1}$$

Hence, we see that the last term transforms like a tensor, but the first term does not because there is a derivative of one of those coordinate transformation factors (hence, a second derivative term). If we can add something to cancel the offending term, we can have a tensor.

One notes that the Christoffel symbol that we defined in the lecture on equivalence principle has a similar transformation property under coordinate transformations:

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\lambda}\frac{\partial^2 \xi^\lambda}{\partial x^\alpha \partial x^\beta}$$

$$\begin{aligned}\Gamma^\mu_{\alpha\beta} \rightarrow \frac{\partial x'^\mu}{\partial \xi^\lambda}\frac{\partial^2 \xi^\lambda}{\partial x'^\alpha \partial x'^\beta} &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial \xi^\lambda}\frac{\partial x^\phi}{\partial x'^\alpha}\frac{\partial}{\partial x^\phi}[\frac{\partial x^\theta}{\partial x'^\beta}\frac{\partial \xi^\lambda}{\partial x^\theta}] \\ &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial \xi^\lambda}\frac{\partial x^\phi}{\partial x'^\alpha}[\frac{\partial^2 x^\theta}{\partial x^\phi \partial x'^\beta}\frac{\partial \xi^\lambda}{\partial x^\theta} + \frac{\partial x^\theta}{\partial x'^\beta}\frac{\partial^2 \xi^\lambda}{\partial x^\phi \partial x^\theta}] \\ &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\alpha}[\frac{\partial^2 x^\theta}{\partial x^\phi \partial x'^\beta}\frac{\partial x^\gamma}{\partial \xi^\lambda}\frac{\partial \xi^\lambda}{\partial x^\theta} + \frac{\partial x^\theta}{\partial x'^\beta}\frac{\partial x^\gamma}{\partial \xi^\lambda}\frac{\partial^2 \xi^\lambda}{\partial x^\phi \partial x^\theta}] \\ &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\alpha}[\frac{\partial^2 x^\gamma}{\partial x^\phi \partial x'^\beta} + \frac{\partial x^\theta}{\partial x'^\beta}\Gamma^\gamma_{\phi\theta}] \\ &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\alpha \partial x'^\beta} + \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\alpha}\frac{\partial x^\theta}{\partial x'^\beta}\Gamma^\gamma_{\phi\theta}\end{aligned}$$

Hence, we see that $\Gamma^\mu_{\alpha\beta}$ also has an inhomogeneous piece which does not transform like a tensor (although the last term does).

Now consider how $\Gamma^\nu_{\mu\beta}V^\beta$ transforms under coordinate transformations:

$$\begin{aligned}
\Gamma^\nu_{\mu\beta}V^\beta \quad \rightarrow \quad & [\frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\beta} + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\frac{\partial x^\theta}{\partial x'^\beta}\Gamma^\gamma_{\phi\theta}]\frac{\partial x'^\beta}{\partial x^\psi}V^\psi \\
&= \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\psi}V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\frac{\partial x^\theta}{\partial x'^\beta}\Gamma^\gamma_{\phi\theta}\frac{\partial x'^\beta}{\partial x^\psi}V^\psi \\
&= \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial}{\partial x'^\beta}[\frac{\partial x^\gamma}{\partial x'^\mu}]\frac{\partial x'^\beta}{\partial x^\psi}V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\Gamma^\gamma_{\phi\theta}V^\theta \\
&= \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial}{\partial x^\psi}[\frac{\partial x^\gamma}{\partial x'^\mu}]V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\Gamma^\gamma_{\phi\theta}V^\theta \\
&= \frac{\partial}{\partial x^\psi}[\frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\mu}]V^\psi - \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial}{\partial x^\psi}[\frac{\partial x'^\nu}{\partial x^\gamma}]V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\Gamma^\gamma_{\phi\theta}V^\theta \\
&= \frac{\partial}{\partial x^\psi}[\delta^\nu{}_\mu]V^\psi - \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\psi \partial x^\gamma}V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\Gamma^\gamma_{\phi\theta}V^\theta \\
&= -\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\psi \partial x^\gamma}V^\psi + \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}\Gamma^\gamma_{\phi\theta}V^\theta
\end{aligned}$$

Hence, we see that the first term of the last line would cancel the first term of Eq. (1) if they were added. In other words, we have found

$$\partial_\mu V^\nu + \Gamma^\nu_{\mu\beta}V^\beta \rightarrow \frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\phi}{\partial x'^\mu}[\partial_\phi V^\gamma + \Gamma^\gamma_{\phi\beta}V^\beta]$$

which is a tensor transformation law. The generalized derivative operation which transforms as a tensor is called a covariant derivative:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\beta}V^\beta.$$

Similarly one can show that

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\beta_{\mu\nu}V_\beta$$

transforms like a tensor. Note the minus sign for the Γ "acting" on the lower index.

2 General rule

In general, the covariant derivative of a tensor gives another tensor, and it can be written as

$$\begin{aligned}
\nabla_\mu V^{\nu_1\nu_2\ldots}{}_{\alpha_1\alpha_2\ldots} \quad = \quad & \partial_\mu V^{\nu_1\nu_2\ldots}{}_{\alpha_1\alpha_2\ldots} + \Gamma^{\nu_1}_{\mu\beta}V^{\beta\nu_2\ldots}{}_{\alpha_1\alpha_2\ldots} + \Gamma^{\nu_2}_{\mu\beta}V^{\nu_1\beta\ldots}{}_{\alpha_1\alpha_2\ldots} + \ldots \\
& -\Gamma^\beta_{\mu\alpha_1}V^{\nu_1\nu_2\ldots}{}_{\beta\alpha_2\ldots} - \Gamma^\beta_{\mu\alpha_2}V^{\nu_1\nu_2\ldots}{}_{\alpha_1\beta\ldots} - \ldots
\end{aligned}$$

Note that the gravitational force equation that we wrote previously can be written succintly as

$$U^\mu \nabla_\mu U^\nu = 0 \tag{2}$$

where

$$U^\mu \equiv \frac{dx^\mu}{d\tau}$$

is the 4-velocity since

$$\begin{aligned}
U^\mu \nabla_\mu U^\nu \quad &= \quad \frac{dx^\mu}{d\tau}[\partial_\mu U^\nu + \Gamma^\nu_{\mu\beta}U^\beta] \\
&= \quad \frac{dx^\mu}{d\tau}[\partial_\mu \frac{dx^\nu}{d\tau} + \Gamma^\nu_{\mu\beta}\frac{dx^\beta}{d\tau}] \\
&= \quad \frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\beta}\frac{dx^\mu}{d\tau}\frac{dx^\beta}{d\tau}.
\end{aligned}$$

Eq. (2) is called the **geodesic equation**. As we will later see, curves x^ν satisfying the geodesic equation has the interpretation of "straight lines" on curved surfaces.