

Lec 12: Riemann Tensor

March 21, 2006

1 Measuring curvature

We saw in the last lecture that when a tangent vector was parallel transported around a closed curve on a 2-sphere, the final vector was not parallel to the original vector. Riemann tensor measures the failure of the vector to return to itself after being parallel transported around a closed path. To see this, let us parallelly transporting a vector around the path ABCD shown in Figure 1.

Consider the parallel transport of the vector from point A to point B . By the definition of parallel transport, we can write

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma_{1\mu}^\alpha V^\mu$$

along $x_2 = b$ curve from point $x^1 = a$ to $x^1 = a + \delta a$. Hence, we find the change to be

$$\begin{aligned} V^\alpha(B) &= V^\alpha(A) + \int_a^{a+\delta a} dx^1 \frac{\partial V^\alpha}{\partial x^1} \Big|_{x^2=b} \\ &= V^\alpha(A) - \int_a^{a+\delta a} dx^1 \Gamma_{1\mu}^\alpha V^\mu \Big|_{x^2=b} \end{aligned}$$

Similar transport from B to C is written as

$$V^\alpha(C) = V^\alpha(B) - \int_b^{b+\delta b} dx^2 \Gamma_{2\mu}^\alpha V^\mu \Big|_{x^1=a+\delta a}$$

Now, from the point C to D , we must integrate “backwards” from point $a + \delta a$ to a :

$$V^\alpha(D) = V^\alpha(C) - \int_{a+\delta a}^a dx^1 \Gamma_{1\mu}^\alpha V^\mu \Big|_{x^2=b+\delta b}$$

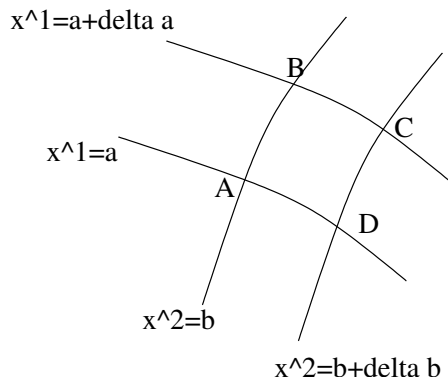


Figure 1: Parallel transport path ABCD.

Finally, from D to A , we write

$$V^\alpha(A)_{final} = V^\alpha(D) - \int_{b+\delta b}^b dx^2 \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a}$$

Putting all these together, we write

$$\begin{aligned} \delta V^\alpha &\equiv V^\alpha(A)_{final} - V^\alpha(A) \\ &= V^\alpha(D) - \int_{b+\delta b}^b dx^2 \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a} - V^\alpha(A) \\ &= V^\alpha(A) - \int_a^{a+\delta a} dx^1 \Gamma_{1\mu}^\alpha V^\mu|_{x^2=b} \\ &\quad - \int_b^{b+\delta b} dx^2 \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a+\delta a} - \int_{a+\delta a}^a dx^1 \Gamma_{1\mu}^\alpha V^\mu|_{x^2=b+\delta b} \\ &\quad - \int_{b+\delta b}^b dx^2 \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a} - V^\alpha(A) \end{aligned}$$

Now, we Taylor expand each integrand to first order:

$$\begin{aligned} \delta V^\alpha &= - \int_a^{a+\delta a} dx^1 [\Gamma_{1\mu}^\alpha V^\mu|_{x^2=b, x^1=a} + \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a}(x^1 - a)] \\ &\quad - \int_b^{b+\delta b} dx^2 [\Gamma_{2\mu}^\alpha V^\mu|_{x^1=a+\delta a, x^2=b} + \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a+\delta a, x^2=b}(x^2 - b)] \\ &\quad - \int_{a+\delta a}^a dx^1 [\Gamma_{1\mu}^\alpha V^\mu|_{x^2=b+\delta b, x^1=a} + \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b+\delta b, x^1=a}(x^1 - a)] \\ &\quad - \int_{b+\delta b}^b dx^2 [\Gamma_{2\mu}^\alpha V^\mu|_{x^1=a, x^2=b} + \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b}(x^2 - b)] \\ &= -\delta a \Gamma_{1\mu}^\alpha V^\mu|_{x^2=b, x^1=a} - \frac{1}{2}(\delta a)^2 \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a} \\ &\quad - \delta b \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a+\delta a, x^2=b} - \frac{1}{2}(\delta b)^2 \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a+\delta a, x^2=b} \\ &\quad + \delta a \Gamma_{1\mu}^\alpha V^\mu|_{x^2=b+\delta b, x^1=a} + \frac{1}{2}(\delta a)^2 \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b+\delta b, x^1=a} \\ &\quad + \delta b \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a, x^2=b} + \frac{1}{2}(\delta b)^2 \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b} \end{aligned}$$

Now, express the terms evaluated at $\dots + \delta\dots$ as that evaluated at \dots :

$$\begin{aligned} \delta V^\alpha &= -\delta a \Gamma_{1\mu}^\alpha V^\mu|_{x^2=b, x^1=a} - \frac{1}{2}(\delta a)^2 \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a} \\ &\quad - \delta b [\Gamma_{2\mu}^\alpha V^\mu|_{x^1=a, x^2=b} + \partial_{x^1}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b} \delta a] \\ &\quad - \frac{1}{2}(\delta b)^2 [\partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b} + \delta a \partial_{x^1} \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b}] \\ &\quad + \delta a [\Gamma_{1\mu}^\alpha V^\mu|_{x^2=b, x^1=a} + \delta b \partial_{x^2}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a}] \\ &\quad + \frac{1}{2}(\delta a)^2 [\partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a} + \delta b \partial_{x^2} \partial_{x^1}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a}] \\ &\quad + \delta b \Gamma_{2\mu}^\alpha V^\mu|_{x^1=a, x^2=b} + \frac{1}{2}(\delta b)^2 \partial_{x^2}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b} \end{aligned}$$

Noticing all the cancellation, we write

$$\begin{aligned}\delta V^\alpha &= [-\partial_{x^1}(\Gamma_{2\mu}^\alpha V^\mu)|_{x^1=a, x^2=b} + \partial_{x^2}(\Gamma_{1\mu}^\alpha V^\mu)|_{x^2=b, x^1=a}] \delta a \delta b \\ &= [\Gamma_{1\mu, 2}^\alpha V^\mu|_{x^2=b, x^1=a} - \Gamma_{2\mu, 1}^\alpha V^\mu|_{x^1=a, x^2=b} + \Gamma_{1\mu}^\alpha V_{, 2}^\mu|_{x^2=b, x^1=a} - \Gamma_{2\mu}^\alpha V_{, 1}^\mu|_{x^1=a, x^2=b}] \delta a \delta b\end{aligned}$$

We now use the fact that the vectors were parallel transported:

$$\delta V^\alpha = [\Gamma_{1\mu, 2}^\alpha V^\mu|_{x^2=b, x^1=a} - \Gamma_{2\mu, 1}^\alpha V^\mu|_{x^1=a, x^2=b} - \Gamma_{1\mu}^\alpha \Gamma_{2\lambda}^\mu V^\lambda|_{x^2=b, x^1=a} + \Gamma_{2\mu}^\alpha \Gamma_{1\lambda}^\mu V^\lambda|_{x^1=a, x^2=b}] \delta a \delta b$$

Factoring out V^μ , we find

$$\begin{aligned}\delta V^\alpha &= [\Gamma_{1\mu, 2}^\alpha - \Gamma_{2\mu, 1}^\alpha + \Gamma_{2\gamma}^\alpha \Gamma_{1\mu}^\gamma - \Gamma_{1\gamma}^\alpha \Gamma_{2\mu}^\gamma] V^\mu|_{x^2=b, x^1=a} \delta a \delta b \\ &= R_{12\mu}^\alpha V^\mu|_{x^2=b, x^1=a} \delta a \delta b\end{aligned}$$

where we have substituted R for the combination of Γ 's. Clearly this is valid for arbitrary indices besides 1 and 2.

2 Properties

There are some simple properties of the Riemann tensor that one can easily check:

1. $R_{\gamma\lambda\mu}^\alpha = -R_{\lambda\gamma\mu}^\alpha$
2. $R_{[\gamma\lambda\mu]}^\alpha = 0$
3. $R_{\gamma\lambda\mu\alpha} = -R_{\gamma\lambda\alpha\mu}$
4. $\nabla_{[\nu} R_{\gamma\lambda]\mu}^\alpha = 0$
5. $R_{\gamma\lambda\mu\alpha} = R_{\mu\alpha\gamma\lambda}$

Here, we have used the notation

$$T_{[\alpha_1 \dots \alpha_n]} = \frac{1}{n!} \sum T_{\text{antisymmeterized}}$$

exercise

Write out $R_{[\gamma\lambda]\mu}^\alpha$ in terms of $R_{\gamma\lambda\mu}^\alpha$ and $R_{\lambda\gamma\mu}^\alpha$.

exercise

Similarly as above, write out $R_{[\gamma\lambda\mu]}^\alpha$ in terms of $R_{\gamma\lambda\mu}^\alpha$, $R_{\lambda\gamma\mu}^\alpha$, $R_{\mu\gamma\lambda}^\alpha$, ...

exercise

Show that

$$R_{[\gamma\lambda\mu]}^\alpha = 0$$