Lec 13: Energy-Momentum Tensor

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1 What is energy density?

We want to eventually use what we learned in the last lecture (Riemann tensor) to write a **tensorial equation** which represents a new gravitational law (i.e. differential equation governing the potential $g_{\mu\nu}$) subsuming Newton's gravitational law. As we have seen in homework 2, Newton's gravitational law as a differential equation is given by

$$\nabla^2\phi(t,\vec{x}) = 4\pi\rho(t,\vec{x}) \tag{1}$$

where $\rho(t,\vec{x})$ is the mass density (or more precisely, energy density since "mass = energy" **relativistically**) which sources the gravitational field $-\vec{\nabla}\phi$. We have also learned in Lecutre 8 that if the metric is perturbed about the flat Minkowski space as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{2}$$

the geodesic equation (free fall equation) gives rise to the Newton's gravitational law for **nonrelativistic situations** with a static source if we identify

$$h_{00}(t,\vec{x}) = -2\phi(\vec{x}).$$

This means that whatever the new general relativistic tensorial gravitational law governing $g_{\mu\nu}$ is, it must reduce to Eq. (1) in the nonrelativitic limit with the ansatz of Eq. (2). In that case, the energy density $\rho(t,\vec{x})$ must be associated with some property of tensor. In this lecture, we would like to understand how $\rho(t,\vec{x})$ fits into the tensorial picture. The part of the book that complements this lecture is section 10.4.

One simple guess is to think that $\rho(t,\vec{x})$ being a component of a tensor. To have an intuition for how one might be able to guess the answer, consider the intuition of Lorentz transforming an energy density. (Because of the equivalence principle, we can almost always start from tensors in Minkowski space.) The numerator of

$$\rho = \frac{\text{energy}}{\text{volume}}$$

is known to transform under Lorentz transformations since we know

$$p^\mu = (\text{energy}, \text{momentum})$$

transforms like a vector under Lorentz transformations. We also know that the denominator transforms under Loretnz transformations because of length contraction. Furthermore, we know from intuition of electromagnetic current that 1/volume (proportional to charge density) can increase by γ due to length contraction while energy can also increase by being multiplied by γ factor under Lorentz transformation, we see that the transformations will not cancel but can be multiplied by two factors of γ under Lorentz transformations. Therefore, we arrive at the important conclusion:

ρ cannot belong to a rank 0 or 1 tensor.

The lowest rank tensor one can then write down consistent with what we have been saying is a rank 2 tensor.

By trial and error (or by constructing explicit examples), one can show that the following definition of components of matrix indeed is a rank 2 tensor.

$$T^{\mu 0} = \text{density of } p^{\mu}$$

$$\begin{aligned} T^{\mu j} &= j\text{th component } p^{\mu} \text{ current density} \\ &= p^{\mu} \text{ flux in the } j\text{th direction} \end{aligned}$$

Note that current density = flux = quantity/(unit time)/(unit cross sectional area).

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exercise

Suppose you are told that $T^{31} = C$ along the y-z plane "wall" which has one corner at the origin and the other corner at $(0, L, L)$. What is the rate of change in the $\hat{z}$ component of the total momentum in the cube (with one corner at the origin and another corner at $(-L, L, L)$) on the other side of the "wall" if you are told that there is no momentum flux on all the other walls?

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exercise

What is the interpretation of T^{00}?

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exercise

Show that $T^{0i} = T^{i0}$.

answer

Since T^{i0} =density of p^i, we can write for mechanical systems

$$T^{i0} = \frac{dm}{dV}(x)\frac{v^i(x)}{\sqrt{1-\vec{v}^2(x)}}$$

where $dm(x)/dV$ represents the mass density at point x. On the other hand

$$\begin{aligned} T^{0i} &= \text{energy flux in the } i\text{th direction} = \frac{dE}{dt dA_i} \\ &= \frac{1}{\sqrt{1-\vec{v}^2(x)}} \frac{\frac{dm}{dV}(x) dV}{dt dA_i} \\ &= \frac{1}{\sqrt{1-\vec{v}^2(x)}} \frac{\frac{dm}{dV}(x) dA_i dl_i}{dt dA_i} \\ &= \frac{dm}{dV}(x)\frac{v^i(x)}{\sqrt{1-\vec{v}^2(x)}} = T^{i0}. \end{aligned}$$

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Similarly, it can be shown from consideration of angular momentum conservation that $T^{ij} = T^{ji}$. Hence, we have the important property

$$T^{\mu\nu} = T^{\nu\mu}.$$

Perhaps the most important property of stress energy tensor is that it is "conserved" in the following sense. In Minkowski space, energy conservation can be written as

$$\begin{aligned}\partial_0 \int d^3x T^{00} &= -\int d^2x n_i T^{0i} \\ &= -\int d^3x \partial_i T^{0i} \\ &= -\int d^3x \partial_i T^{i0}\end{aligned}$$

Hence, we can write energy conservation as

$$\partial_0 T^{00} + \partial_i T^{i0} = \partial_\mu T^{\mu 0} = 0$$

More generally, energy-momentum conservation can be written as

$$\partial_\mu T^{\mu\nu} = 0.$$

Finally, in curved spacetime, we can use the comma goes to semi-colon rule to write

$$\nabla_\mu T^{\mu\nu} = 0.$$

2 Examples

2.1 Pressureless Dust

A swarm of noninteracting, massive particles is typically referred to as "dust". One of the defining properties of an "ideal dust" is that at any given point, one can go to the rest frame of a group of particles within an infinitesimal volume. Hence, in that frame (called comoving frame) at any given point, we have

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, note that particle velocity field at this point is $U^\mu \equiv \frac{dx^\mu}{d\tau} = (1, 0, 0, 0)$ in this frame. This means

$$T^{\mu\nu} U_\mu U_\nu = \rho.$$

However, this is a coordinate independent statement. (At this point, we can be dealing with general relativistic tensors as well by equivalence principle.) Now, note that $U^\mu U_\mu = -1$ is also a coordinate independent statement. This leads us to guess (or derive starting from the defining property of $T^{\mu\nu}$) that

$$T^{\mu\nu} = \rho U^\mu U^\nu.$$

Hence, note that the dust system is characterized by two functions $\rho(t, \vec{x})$ and $U^\mu(t, \vec{x})$ where $U^\mu U_\mu = -1$.

2.2 Ideal Fluid

The pressureless dust did not exchange momentum with surrounding spatial cells in the comoving frame. However, we know if we have hot gas inside of a box, there must be pressure exerted on the walls of the box. If the space is isotropic, we must then have in the comoving frame

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where P is the pressure. Again, from the properties of U^μ, one can then infer the expression in a generic frame

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + P g^{\mu\nu}$$

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exercise

Evaluate $T^{\mu\nu}U_\mu U_\nu$ for an ideal fluid.

2.3 Electromagnetic Field

The energy-momentum tensor for electromagnetic field is

$$T^{\mu\nu} = g_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$