Lec 16: Conservation Laws and Physics in Curved Spacetime

April 11, 2006

1 Killing Vector

We would like to now explore the physical consequences of the solutions to Einstein's equations. In the last lecture, we solved for the Schwarzschild geometry, which is the gravitational analog of Coulomb's law solution. Just as in electromagnetism, we would like to introduce test particles and see how the test particles move in the presence of the "external" field that we solved for in the last lecture (the Schwarzschild geometry). In classical mechanics, it is well known that conservation laws (such as momentum and energy conservation) help to simply the task of solving Newton's equations of motion. Such tricks are also helpful in general relativity to solve the geodesic equations in situations in which there is a high degree of symmetry. To construct the conserved quantities, it is useful to know quantities called Killing vectors.

The vector ξ^α derived from the following differential equation is called a Killing vector:

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0.$$

Although it has a geometrical meaning, we will concentrate on its basic usage. To construct conserved quantities using this, we observe the following: If ξ^μ is a Killing vector field and $u^\mu = \frac{dx^\mu}{d\lambda}$ is a tangent vector to a geodesic, then

$$\xi^\mu u_\mu$$

is a constant along the geodesic. Since a test particle follows a geodesic, this says that $\xi^\mu u_\mu$ is the conserved quantity that we are looking for. Often, we can then use locally freely falling frame to interpret the conserved quantity.

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exercise

Show that $\xi^\mu u_\mu$ is a conserved quantity along a geodesic.

answer

Along a geodesic, the change in $\xi^\mu u_\mu$ can be expressed as

$$u^\nu \nabla_\nu (\xi^\mu u_\mu) = \xi^\mu u^\nu \nabla_\nu u_\mu + u_\mu u^\nu \nabla_\nu \xi^\mu$$

The first term vanishes because _ _ _ _ _ _ _ _ _ _ _ _ _ and the second term vanishes because _ _ _ _ _ _ _ _ _ _ _ _ _.

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exercise

Show that any constant vector a^μ is a Killing vector in Minkowski space

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exercise

What is the conserved quantity associated with the Killing vector $a^\mu = (1, 0, 0, 0)$ in Minkowski space?

answer

We have

$$a^\mu u_\mu = u_0 = -u^0 = -\frac{dt}{d\lambda}.$$

Now, by definition of timelike geodesic, we have

$$\begin{aligned} u^\mu u_\mu &= -1 \\ &= -(u^0)^2 + \vec{u}^2 \\ &= -(u^0)^2 + (u^0)^2 \vec{v}^2. \end{aligned}$$

Hence, we have found

$$(u^0)^2 = \gamma^2$$

Since

$$E = m\gamma,$$

we have found that the conserved quantity is

$$a^\mu u_\mu = -\frac{E}{m}$$

along a geodesic in Minkowski space. This is of course obvious given that there is no force acting on the particle.

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exercise

What is the conserved quantity associated with the Killing vector $a^\mu = (0, 1, 0, 0)$ in Minkowski space?

answer

We have

$$a^\mu u_\mu = u_1 = u^1 = \frac{dx}{d\lambda} = \frac{dt}{d\lambda}\frac{dx}{dt}$$

From the previous exercise, we learned that

$$\frac{dt}{d\lambda} = \frac{E}{m}$$

Hence, we see that the conserved quantity in this exercise is

$$a^\mu u_\mu = \gamma v_x = \frac{p_x}{m}.$$

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2 Gravitational Redshift in Schwarzschild

In Minkowski space, plane waves of light have the wave form

$$\vec{E} = \vec{E}_0 \Re e^{ik_\mu x^\mu}$$

where k^μ is a tangent vector to a null geodesic, i.e.

$$k^\mu k_\mu = 0 \tag{1}$$

with k_0 having the interpretation of $-E$. Expressed in a frame independent manner, we can write

$$E = -k_\mu u^\mu$$

where u^μ is the 4-velocity associated with the locally freely falling observer making the observation of the energy. (Note that in the Minkowski frame, we have $E = -k_0$ by construction.) In the Schwarzschild frame, an observer at a fixed radius has the 4-velocity component $(u^0, 0, 0, 0)$ satisfying the normalization

$$u^\mu u_\mu = -1 = (u^0)^2 g_{00} = -(u^0)^2 (1 - \frac{2M}{r}).$$

This implies

$$u^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

and

$$E = -k_0 \frac{1}{\sqrt{1 - \frac{2M}{r}}}. \tag{2}$$

Now, to see what quantity is conserved for a photon travelling in the Schwarzschild metric, we need to find the Killing vector in Schwarzschild metric? Suppose we guess that

$$\xi_\alpha = (f(r), 0, 0, 0)$$

since the metric is spherically symmetric and the Killing vector that we desire is associated with time translations. We find the equation to be

$$\begin{aligned}
\xi_{0;\beta} + \xi_{\beta;0} &= f_{,\beta} - \Gamma^\lambda_{0\beta}\xi_\lambda + 0 - \Gamma^\lambda_{\beta 0}\xi_\lambda \\
&= f_{,\beta} - 2\Gamma^0_{0\beta} f(r) \\
&= \delta_{\beta 1}[f_{,1} - (\frac{2M}{r^2}\frac{1}{1 - \frac{2M}{r}})f] \\
&= 0.
\end{aligned}$$

We thus find

$$f \propto 1 - \frac{2M}{r}.$$

Hence, the Killing vector

$$\xi^\mu = (1, 0, 0, 0)$$

yields the conserved quantity

$$C \equiv n_\mu \xi^\mu = n_0$$

to be the conserved quantity where n^μ is the null vector associated with the photon. Since n^μ is tangent to a null geodesic, we have from Eq. (1)

$$n^\mu \propto k^\mu.$$

Hence, the energy of a photon at two locations $r_1$and r_2 can be compared with k_0 being a constant in Eq. (2):

$$\frac{E(r_1)}{E(r_2)} = \frac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}}.$$

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exercise

Evaluate $E(r_1)/E(r_2)$ for $r_{1,2} \gg 2M$ and interpret the result in terms of gravitational energy.
