# Lec 17: Orbits Around Schwarzschild

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## 1 Special Killing Vectors

Suppose we have $\xi^\lambda = \text{const}$. We have $\xi_\alpha = g_{\alpha\lambda}\xi^\lambda$

$$\begin{aligned} \xi_{\alpha,\beta} + \xi_{\beta,\alpha} &= 2\Gamma^\lambda_{\alpha\beta}\xi_\lambda \\ (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha})\xi^\lambda &= \xi_\lambda g^{\lambda\mu}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \\ &= \xi^\mu(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \end{aligned}$$

This implies

$$\xi^\mu g_{\alpha\beta,\mu} = 0.$$

In particular, if we have $g_{\alpha\beta}$ independent of any coordinate $\mu$, we have $\xi^\alpha = \delta^{\alpha\mu}$ being a Killing vector. In that case, we have

$$\xi^\alpha u_\alpha = \frac{dx_\mu}{d\tau}$$

being conserved along a geodesic.

## 2 Application to Schwarzschild

Consider the Schwarzschild metric:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The time independence of the metric says that we have

$$\tilde{E} \equiv -\frac{dx_0}{d\lambda} = \text{const}.$$

The quantity

$$\frac{dx^0}{d\lambda} = -\tilde{E}g^{00} = \frac{\tilde{E}}{1 - \frac{2M}{r}}$$

has the interpretation of energy per unit mass. The $\phi$ independence gives

$$\tilde{L} \equiv \frac{dx_3}{d\lambda} = \text{const}. \tag{1}$$

The quantity

$$\frac{dx^3}{d\lambda} = \tilde{L}g^{33} = \frac{\tilde{L}}{r^2\sin^2\theta}$$

which has the interpretation of angular velocity (which means $\tilde{L}$ has the interpretation of angular momentum per unit mass). We will now conveniently choose the coordinate such that

$$\theta = \pi/2.$$

(You will see this in your homework.)

The timelike (lightlike) nature of the geodesic means

$$\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda}g^{\mu\nu} = s \tag{2}$$

where $s = -1$ for timelike and $s = 0$ for lightlike geodesics. This implies

$$\tilde{E}^2 g^{00} + (\frac{dr}{d\lambda})^2 g_{11} + \tilde{L}^2 g^{33} = s$$

$$g_{11}[-\tilde{E}^2 + (\frac{dr}{d\lambda})^2] + \tilde{L}^2 g^{33} = s$$

$$(\frac{dr}{d\lambda})^2 = \tilde{E}^2 + s g^{11} - \tilde{L}^2 \frac{g^{33}}{g_{11}}$$

$$(\frac{dr}{d\lambda})^2 = \tilde{E}^2 - (1 - \frac{2M}{r})(-s + \frac{\tilde{L}^2}{r^2}) \tag{3}$$

Note that

$$\tilde{U} \equiv (1 - \frac{2M}{r})(-s + \frac{\tilde{L}^2}{r^2})$$

can be interpreted as a kind of potential for one dimensional motion since $(\frac{dr}{d\lambda})^2$ looks like a kinetic energy. However, this is just an analogy since the motion is not really one dimensional.

We can make the differential equation to be dependent on $\phi$ instead of $\lambda$ by first noting from Eq. (3)

$$(\frac{dr}{d\lambda})^2 = (\frac{dr}{d\phi})^2(\frac{d\phi}{d\lambda})^2 = \tilde{E}^2 - \tilde{U}.$$

Since from Eq. (1), we have

$$\frac{d\phi}{d\lambda} = \frac{\tilde{L}}{g_{33}|_{\theta=\pi/2}} = \frac{\tilde{L}}{r^2},$$

we find

$$(\frac{dr}{d\phi})^2 = \frac{\tilde{E}^2 - \tilde{U}}{\tilde{L}^2/r^4}. \tag{4}$$

To identify the location of the circular orbits, one solves

$$\frac{d}{dr}\tilde{U} = 0.$$

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**Exercise**

Why is this true?

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The solution is simply

$$r_c = \frac{-\tilde{L}^2 \pm \tilde{L}^2\sqrt{1 + \frac{12M^2 s}{\tilde{L}^2}}}{2Ms}. \tag{5}$$

For $s < 0$, we have to take the "-" root to obtain a positive radius, while for $s > 0$ we have to take the "+" root.

## 2.1 Perihelion of Mercury (Box 6.4 of your book – pg. 100)

Consider a timelike geodesic: $s = -1$ in Eq. (2). Let

$$u \equiv \frac{1}{r}.$$

The differential Eq. (4) turns into

$$(\frac{du}{d\phi})^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - (1 - 2Mu)(\frac{1}{\tilde{L}^2} + u^2). \tag{6}$$

It turns out that this equation differs from the Newtonian version only because of the $\mathcal{O}(u^3)$ term.

We would like to solve this using perturbation series:

$$u = u_0 + \lambda u_1 + \mathcal{O}(\lambda^2)$$

where $\lambda$ is the formal expansion parameter and $\mathcal{O}(u^3) = \mathcal{O}(\lambda)$. To obtain a stable perturbation series, it is convenient to use the differentiated (with respect to $\phi$) version of Eq. (6). We find

$$2(\frac{du}{d\phi})\frac{d^2u}{d\phi^2} = 2M\frac{du}{d\phi}(\frac{1}{\tilde{L}^2} + u^2) + -(1 - 2Mu)(2u)\frac{du}{d\phi}.$$

Assuming $\frac{du}{d\phi} \neq 0$, we find

$$\frac{d^2u}{d\phi^2} = \frac{M}{\tilde{L}^2} + -u + 3Mu^2.$$

Now, assume $u \ll 1$ (nonrelativistic limit) and thereby treat $3Mu^2$ as a perturbation. We find

$$u_0 = \frac{1}{2}B + C\cos\phi$$

where

$$B = \frac{2M}{\tilde{L}^2}.$$

We can compute $C$ by putting this back into Eq. (6).

$$C^2 = \frac{M^2}{\tilde{L}^4} + \frac{\tilde{E}^2 - 1}{\tilde{L}^2}.$$

To order $\lambda$, we find

$$\frac{d^2u_1}{d\phi^2} + u_1 \quad = \quad 3Mu_0^2.$$

As you will show in your homework, to linear order in $C$, the perturbation turns out to be

$$u_1 = A + X\phi\sin\phi + Y\cos\phi$$

with $\{A, X, Y\}$ functions of $\{M, B, C\}$. Hence, as you will show in your homework, in the nonrelativistic, nearly circular orbit limit, you will find

$$r \approx \frac{1}{u_0 + u_1} \approx \frac{2/B/(1+\Delta)}{1 + \frac{2C}{B}\frac{1}{1+\Delta}\cos[(1 - \frac{3MB}{2})\phi]}$$

where

$$\frac{3}{2}MB = \frac{3M^2}{\tilde{L}^2} \ll 1$$

and $\Delta$ is a constant that you will find in your homework.

The closest approach is at $\phi = 0$ when $\cos[(1 - \frac{3MB}{2})\phi] = 1$. However, the next closest approach is not at $\phi = 2\pi$, but

$$\phi \approx 2\pi(1 + \frac{3M^2}{\tilde{L}^2}).$$

Let's plug in numbers to see the magnitude of the angular shift per orbital period of $T \approx 0.24$ yr for Mercury. The solar mass is given in geometrical units as$M = 1.47$km and the mercury orbit radius is given as $r_c \approx 5.55 \times 10^7$km. Using Eq. (5), we have for $s = -1$

$$r \approx r_c \approx \frac{\tilde{L}^2}{M}$$

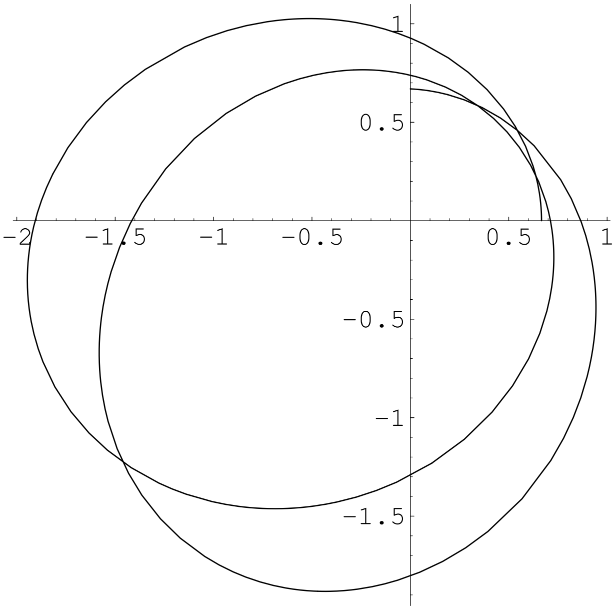
and hence

$$\begin{aligned} \frac{\delta\phi}{T} &= \frac{6\pi M^2}{\tilde{L}^2 T} \\ &= \frac{6\pi M}{r_c T} \\ &\approx \frac{6\pi 1.47\text{km}}{5.55 \times 10^7 (0.24\text{yr})} \\ &= 2.1 \times 10^{-4}\text{rad/century} \end{aligned}$$

which corresponds to 43"/century.

To qualitatively visualize the trajectory, consider the following plot of the function

$$r = \frac{1}{1 + 0.5\cos(0.9\phi)}$$



plotted from $\phi = 0$ to $\phi = 4.5\pi$: