Lec 18: Bending of Light and Black Holes

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1 Bending of Light

Another impressively dramatic GR effect that has been experimentally confirmed is the bending of light by gravitational field. From the last lecture, we learn that for null (lightlike) worldlines, we have

$$\tilde{U} = (1 - \frac{2M}{r})\frac{\tilde{L}^2}{r^2}$$

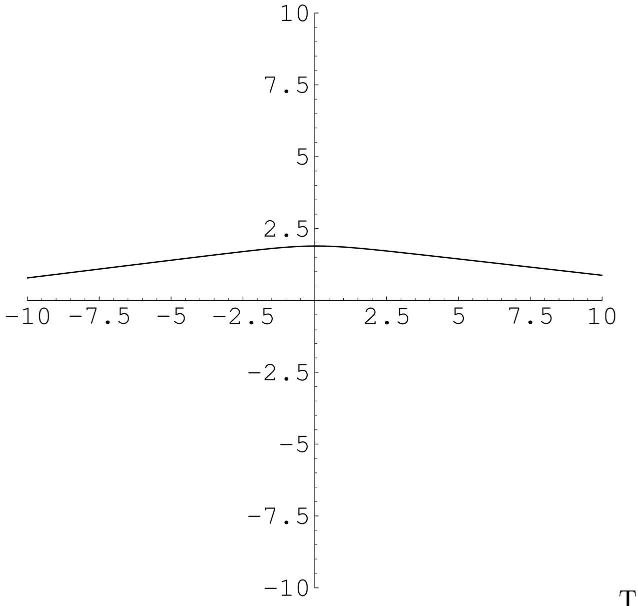
and

$$(\frac{du}{d\phi})^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - (1 - 2Mu)u^2 \tag{1}$$

where $u = 1/r$ as before. Again, it proves useful to work with the differentiated version of this equation:

$$\frac{d^2u}{d\phi^2} = 3Mu^2 - u.$$

Consider the schematic solution of $(\cos\phi, \sin\phi)/u(\phi)$ in the following figure:



The mass is at the origin and the light is initially coming from $\phi = \frac{\delta\phi}{2}$ at $r = \infty$.

We can solve the equation using the perturbation approach again assuming that the deflection is small (treating M as a perturbation of $\mathcal{O}(\lambda)$): $u = u_0 + \lambda u_1 + ...$ The undeflected light travels at a straight line when $M = 0$:

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0$$

$$u_0 = A\sin\phi + B\cos\phi$$

We will set $B = 0$, corresponding to the case in which the light path is a horizontal line in the figure in the absence of mass M. The constant A can be fixed by putting u_0 back into Eq. (1) to zeroth order in λ:

$$A^2 = \frac{\tilde{E}^2}{\tilde{L}^2}.$$

The leading order in perturbation is

$$\frac{d^2u_1}{d\phi^2} = 3Mu_0^2 - u_1$$

giving

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3MA^2\sin^2(\phi) = \frac{3}{2}MA^2[1 - \cos 2\phi].$$

Solving, we find

$$u_1 = C_1\cos\phi + C_2\sin\phi + \frac{3}{2}MA^2 + \frac{A^2M}{2}\cos 2\phi.$$

Since we want the perturbation to be symmetric under $\phi \to \pi - \phi$, we set $C_1 = 0$. We can then set C_2inserting this into $\mathcal{O}(\lambda)$ portion of Eq. (1):

$$Mu_0^3 = u_0u_1 + \frac{du_0}{d\phi}\frac{du_1}{d\phi}.$$

This implies $C_2 = 0$.

Thus, adding this to the u_0 solution, we have arrived at

$$u = A[\sin(\phi) + \frac{3}{2}AM + \frac{AM}{2}\cos(2\phi)].$$

Going back to the picture, we have the deflection angle to be at $\phi = \delta\phi/2$ when $u = 0$:

$$0 = A[\sin(\frac{\delta\phi}{2}) + \frac{3}{2}AM + \frac{AM}{2}\cos(\delta\phi)].$$

In the small angle limit, we can solve for $\delta\phi$ and obtain

$$\delta\phi = 4AM.$$

We can express this in a phenomenologically more transparent manner by noting that the closest approach is at $\phi = \pi/2$:

$$\frac{1}{r_{\text{min}}} = A[1 + AM] = A + \mathcal{O}(\lambda).$$

Hence,

$$\delta\phi = \frac{4M}{r_{\text{min}}} + \mathcal{O}(\lambda^2).$$

2 Black holes (Section 6.4 of your book - pg. 102)

Consider a light geodesic for $r < 2M$. Because the metric becomes

$$ds^2 = |1 - \frac{2M}{r}|dt^2 - \frac{dr^2}{|1 - \frac{2M}{r}|} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

the radial direction becomes timelike with $r = 0$ in the future and the time direction becomes spacelike. Which direction is the future ($\hat{e}_r$ or $-\hat{e}_r$)? Consider a photon falling in. From the last lecture, the radial worldline is

$$\frac{dr}{d\lambda} = -\sqrt{\tilde{E}^2 - (1 - \frac{2M}{r})\frac{\tilde{L}^2}{r^2}}$$

where the minus sign corresponds to the fact that it is infalling for $r > 2M$. As long as $\tilde{E}$ is sufficiently large, the photon trajector can be coming in from infinity all the way to $r = 2M$. After crossing $r = 2M$, the photon worldline continues along this wordline. Hence, forward in the worldline given by positive $d\lambda$ corresponds to a negative dr. Hence, the future for $r < 2M$ is corresponding to r decreasing towards $r = 0$. Even a photon has no choice but to move "forward" in "time"! Because of this direction of lightcone for $r < 2M$, not even a photon can avoid encoutering $r = 0$ region. What happens after any object reaches $r = 0$?

As you will show in your homework, the Riemann tensor invariant

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to \infty$$

at $r = 0$. Since this is a coordinate invariant quantity, we see that the curvature blows up at $r = 0$. One can show that the rate at which two neighboring geodesics approach each other is given by

$$\frac{dx^\alpha}{d\lambda}\nabla_\alpha[\frac{dx^\lambda}{d\lambda}\nabla_\lambda n^\beta] = -R^\beta{}_{\mu\gamma\delta}\frac{dx^\mu}{d\lambda}\frac{dn^\gamma}{d\lambda}\frac{dx^\delta}{d\lambda}$$

which is the covariant version of a force equation with n^β interpreted as a displacement between two neighboring geodesics. In Newtonian language, the rate of geodesic deviation is the tidal force, and the intuitive correspondence between a Newtonian potential ϕ and R_{j0k0} is

$$\frac{\partial^2\phi}{\partial x^j \partial x^k}$$

which measures, rougly speaking, changes in the neighboring contituent's acceleration. This means that the tidal forces diverges at the singularity and any particle will be shredded to pieces.

Since Einstein's equations truly break down at $r = 0$, "life" for any object ends at $r = 0$ if one takes the singularity as something truly physical. Hence, observers outside of the black hole will see something fall in but nothing come out classically. However, quantum field theoretically, one expects: a) there is no true singularity at $r = 0$ due to quantum gravity b) Hawking radiation makes the black hole have only a finite lifetime.

You may be worried at this point that we have used a coordinate system that is singular at $r = 2M$ and $r = 0$ to analyze physics near $r = 2M$. As you will show in your homework, there is a coordinate system which remains perfectly well defined at $r = 2M$ and $r = 0$ (up to the usual spherical coordinate singularity). It can be shown that the results of our analysis remains the same when done properly in this non-singular coordinate system.

There are several other features that you should be able to compute yourself regarding black holes:

a) Photons from near $r = 2M$ has an infinite redshift when escaping to $r = \infty$. That means all photons lose color (black).

b) Photons from $r = 2M + \epsilon$ take an $\mathcal{O}(1/\epsilon)$ amount of Schwarzschild coordinate time to escape to $r = \infty$. (hint: The null geodesic in for motion in the dt direction is

$$\frac{dt}{dr} = \frac{\pm 1}{|1 - \frac{2M}{r}|}.$$

One can integrate this to obtain

$$t - t_0 = \pm[r - r_0 + 2M \ln |\frac{2M - r}{2M - r_0}|].$$

where t_0 and r_0 are integration constants.) This means it takes an infinite amount of time to escape from a point arbitrarily close to $r = 2M$.

c) Any particles take a finite proper time to fall to $r = 2M$ and then eventually to $r = 0$. (Hint: Integrate the proper time using the $dr/d\lambda$ equation.)