

Lecture 19: Cosmology

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1 Introduction to Cosmology (Chapter 7 of your book)

We have seen that general relativity gives small corrections to Newtonian gravity except in the case of black holes where

$$\frac{M}{r} \gtrsim \frac{1}{2}$$

such as near the horizon of a black hole. In the black hole case, this was achieved by making r small for a fixed M . Of course, another way one can achieve this condition is to make M large. This is the realm of cosmology, where we have

$$\frac{M}{R} \sim \frac{\rho R^3}{R} = \rho R^2 \gtrsim 1$$

with ρ being the energy density. Even if ρ is small, we can achieve relativistic effects by making R sufficiently large.

According to the observations of galaxies on the largest scales and the measurement of the cosmic microwave background radiation, the stress energy tensor describing the “fluid” of the “universe” seems to be *homogeneous and isotropic*. Such simplicity also resonates with our natural intuition that we do not occupy a special point in the universe.

The assumption of homogeneity and isotropy implies the following form of the metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right].$$

This metric is called Friedmann-Robertson-Walker metric. Note that although it is simple to see that the metric is homogeneous when $K = 0$, naively, one may conclude that the metric is not homogeneous for $K \neq 0$. However, it is straight forward to show that the following “quasi-translations” keep the metric invariant:

$$\vec{x}' = \vec{x} + \vec{a} \left\{ \sqrt{1 - K\vec{x}^2} - [1 - \sqrt{1 - K\vec{a}^2}] \left(\frac{\vec{x} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right) \right\}$$

where \vec{a} is an arbitrary constant 3-vector.

2 Hubble Law

Suppose two spatially separated galaxies are treated as point particles. Since observations tell us that a typical galaxy is at rest with respect to the homogeneous and isotropic frame (by definition), we can write the physical distance between the two galaxies as

$$D_P = (\Delta r) a(t)$$

where Δr is the coordinate separation of the two galaxies along the radial direction (we have chosen the origin of the spatial coordinates such that the total coordinate separation between the two galaxies is only along the radial direction). Hence, in Minkowski space physics language, there is a relative velocity between any two galaxies

$$v_H = (\Delta r) \dot{a} = D_P \frac{\dot{a}}{a} \equiv D_P H$$

whose magnitude is larger for larger spatial separations. Here, the quantity H is called the Hubble expansion rate.

From the Minkowski intuition, one would expect there to be a redshift of light signal sent from one galaxy to the other galaxy separated by coordinate spatial distance Δr . Just as in computing the gravitational redshift in the Schwarzschild geometry case, we would like to compare

$$w_1 = -(k_\mu U_1^\mu)|_{P_1}$$

and

$$w_2 = -(k_\mu U_2^\mu)|_{P_2}$$

where k^μ is the wave vector of the light wave, U_i^μ are the 4-velocity of geodesic observers at the emission (point P_1) and reception (point P_2). Since both the emitter and the receiver are at fixed spatial comoving coordinates, we have $U_i^\mu = (1, 0, 0, 0)$ and thus

$$w_1 = -k_0|_{P_1}$$

$$w_2 = -k_0|_{P_2}.$$

Now, by definition

$$k^\mu = \frac{dx^\mu}{d\lambda}$$

satisfies the null geodesic equation. We can thus write the geodesic equation as

$$\frac{d}{d\lambda}k^\mu + \Gamma_{\alpha\beta}^\mu k^\alpha k^\beta = 0$$

which for the k^0 component yields

$$\frac{d}{d\lambda}k^0 + \Gamma_{\gamma\delta}^0 k^\gamma k^\delta = 0.$$

As you have shown in your homework, the Christoffel symbol can be written as

$$\Gamma_{\gamma\delta}^0 = \sum_{i,j=1}^3 \delta_{\gamma i} \delta_{\delta j} \left(\frac{\dot{a}}{a}\right) h_{ij}$$

where h_{ij} is the metric on the constant time hypersurface. Hence, the geodesic equation becomes

$$\frac{d}{d\lambda}k^0 + \sum_{i,j=1}^3 \left(\frac{\dot{a}}{a}\right) h_{ij} k^i k^j = 0. \quad (1)$$

Since the geodesic is null, we also know

$$k_\mu k^\mu = 0 = -(k^0)^2 + \sum_{i,j=1}^3 h_{ij} k^i k^j$$

which allows us to rewrite Eq. (1) as

$$\frac{d}{d\lambda}k^0 + \left(\frac{\dot{a}}{a}\right)(k^0)^2 = 0.$$

Dividing through by k^0 gives

$$\frac{d}{dt}k^0 + \left(\frac{\dot{a}}{a}\right)(k^0) = 0.$$

The solution to this equation is simply

$$\int \frac{dk^0}{k^0} = - \int \frac{da}{a}$$

$$(w_1/w_2) = (a(t_2)/a(t_1))$$

Hence, there is a redshift in the frequency as the universe expands. That means that light sent from one galaxy to another will redshift in frequency.

As discussed in Weinberg pages 418-427, there are few classic methods for measuring distances:

1. luminosity distance
2. angular diameter
3. proper motion
4. parallax

Let us take a look at the luminosity distance which gives the strongest evidence for dark energy. Luminosity distance assumes that one knows the absolute magnitude (brightness) L of a light emitting object. Let P be the total power received by the detector. Suppose we treat the energy of light as constituting of photons in a unit cell. Due to the redshift, the energy of light (proportional to the frequency w), the energy of a photon is redshifted by an amount

$$\delta E_0 = \delta E_s \left[\frac{a(t_s)}{a(t_0)} \right].$$

Furthermore, the “first” photon reaches us at time t_0 defined by

$$\int dr f(r) = \int_{t_s}^{t_0} \frac{dt}{a(t)}$$

(where $f \equiv \frac{1}{\sqrt{1-Kr^2}}$) while the “second” photon reaches us at time $t_0 + \delta t_0$

$$\int dr f(r) = \int_{t_s+\delta t_s}^{t_0+\delta t_0} \frac{dt}{a(t)} = \left(\frac{\delta t_0}{a(t_0)} - \frac{\delta t_s}{a(t_s)} \right) + \int_{t_s}^{t_0} \frac{dt}{a(t)}$$

which implies

$$\delta t_0 = \delta t_s \frac{a(t_0)}{a(t_s)}.$$

Since the detector subtending a small area A absorbs a fraction of the emitted energy proportional to the area, we have the measured energy being

$$P\delta t_0 = [L\delta t_s] \left[\frac{a(t_s)}{a(t_0)} \right] \left[\frac{A}{4\pi r^2 a^2(t_0)} \right].$$

This corresponds to

$$P = [L] \left[\frac{a(t_s)}{a(t_0)} \right]^2 \left[\frac{A}{4\pi r^2 a^2(t_0)} \right].$$

The apparent luminosity is defined to be

$$l \equiv \frac{P}{A}.$$

In Euclidean space, this is expected to be

$$l = \frac{L}{4\pi d_L^2}$$

where d_L is called the luminosity distance. Hence

$$\begin{aligned} d_L &= \sqrt{\frac{L}{4\pi l}} \\ &= r \frac{a^2(t_0)}{a(t_s)}. \end{aligned}$$

Now, since the redshift of light is given by

$$\frac{w(t_s)}{w(t_0)} = \frac{a(t_0)}{a(t_s)} \equiv 1 + z,$$

we can write down the relationship between the luminosity distance and the redshift as

$$d_L = r a(t_0) [1 + z].$$

Now, we would like an expression for r as a function of z . Expanding $a(t)$ in Taylor series,

$$a(t) = a(t_0)[1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + \dots],$$

we can write

$$z = H_0(t_0 - t_s) + (1 + \frac{q_0}{2})H_0^2(t_0 - t_s)^2 + \dots$$

which gives $(t_0 - t_s)$ as a function of z . Inverting this we find

$$(t_0 - t_s) = \frac{1}{H_0}[z - (1 + \frac{q_0}{2})z^2 + \dots].$$

We can then use

$$\int \frac{dt}{a(t)} = \int \frac{dr}{\sqrt{1 - Kr^2}}$$

to write

$$\int_{t_s}^{t_0} \frac{dt}{a} = \frac{1}{a(t_0)} \int_{t_s}^{t_0} [1 + H_0(t_0 - t) + (1 + \frac{q_0}{2})H_0^2(t_0 - t)^2 + \dots]dt = r + O(r^2)$$

to solve for r as a function of $(t - t_0)$ and hence as a function of z . Hence, finally, we can write

$$d_L = \frac{1}{H_0}[z + \frac{1}{2}(1 - q_0)z^2 + \dots]. \tag{2}$$

Ideally, one would like to measure something nearby with a distinctive yet categorical peculiar feature X (either in the light spectrum or its time variability) and determine its distance by other means feasible for measuring nearby distances. Then one makes an assumption about the far away objects with the identifiable peculiar feature X to behave the same way as for the close by objects, differing only in distance. Hence, by measuring luminosity and the redshift, we can confirm Eq. (2). This is what Edwin Hubble did to give rise to the birth of modern observational cosmology.