Lecture 22: Intro to Gravity Waves

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1 Gravity is a more complicated version of Maxwell's equations

In lecture 14, when you were learning about obtaining the factor of 8π in front of the $T_{\mu\nu}$ for Einstein's equations, you learned about the perturbation of what Einstein's equation look like for the linear order pertubation of $g_{\alpha\beta}$ written as

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x)$$

where $|h_{\alpha\beta}| \ll 1$ and $T_{\mu\nu}$ is treated as a perturbation (which is implicit in the fact that the zeroth order solution is Minkowski space). We found back then that

$$R_{\mu\nu} = \frac{1}{2}(h_{\mu\alpha,\nu}{}^{;\alpha} + h_{\nu\alpha,\mu}{}^{;\alpha} - h_{\mu\nu,\alpha}{}^{;\alpha} - h^{\alpha}_{\alpha,\mu\nu}).$$

Using this, we can write the linearized Einstein's equations as

$$\frac{-1}{2}\partial^{\lambda}\partial_{\lambda}\bar{h}_{\alpha\beta} + \partial^{\lambda}\partial_{(\beta}\bar{h}_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^{\lambda}\partial^{\gamma}\bar{h}_{\lambda\gamma} = -8\pi T_{\alpha\beta} \quad (1)$$

where

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h^{\lambda}{}_{\lambda}.$$

(Recall that ( ) symbol means symmeterized with respect to the indices with a factor of $1/N! = 1/2$ in this case.) The remarkable feature is that this looks very much like Maxwell's equations written in terms of vector potential (see for example lecture 2)

$$\partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = -j^{\nu}.$$

Hence, one can employ the usual techniques of solving Maxwell's equations to solve linearized Einstein's equations as well.

2 Importance of Quadrupole moment

As you can read about in pages 252-253 in your book, there is a way to choose the gauge (just like in elctromagnetism) to write Eq. (1) in the transverse gauge (sometimes also called the Lorentz gauge) as

$$\partial^{\lambda}\partial_{\lambda}\bar{h}_{\alpha\beta} = 16\pi T_{\alpha\beta}.$$

This equation can be solved just as in electromagnetism (see lecture 2 or page 253) as

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4\int d^3x' \frac{T_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}.$$

As you can read about on pages 262-263, you can write down the solution for h_{ij} (spatial components $i, j \in \{1,2,3\}$, as one can choose a particular gauge in which $h_{00} = 0$) to leading approximation (in the limit that $T_{\mu\nu}$ is very slowly varying in spacetime and the observer is very far away from the source) as

$$h_{ij}(t,\vec{r}) \approx \frac{2}{r}\ddot{I}_{ij}(t-r)$$

where

$$I_{ij}(t) = \int d^3x \rho(t, \vec{x}) x_i x_j$$

and r is the distance to the source. As you may recall from electromagnetism, I_{ij} is the quadrupole moment (up to the subtraction of the trace part, which also does not contribute to the radiation as can be seen simply in the traceless gauge). Hence, unlike electromagnetism for which the leading contribution to radiation comes from dipole distribution of the source, in gravity, the leading contribution to radiation comes from the quadrupole moment of the source.