# Derivation of Lorentz Transformations

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In 1D, start with the general coordinate transformations

$$\Lambda(v)\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} a(v) & b(v) \\ c(v) & d(v) \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

where the primed frame is moving with respect to the unprimed frame with constant velocity $v$. Now, consider an event in the unprimed frame of measuring an object at rest at time $t = T$. Since by definition of relativity, the object is moving leftward at constant speed $v$ for some time $\tau$ in the primed coordinate, we have

$$\begin{pmatrix} a(v) & b(v) \\ c(v) & d(v) \end{pmatrix}\begin{pmatrix} T \\ 0 \end{pmatrix} = \begin{pmatrix} \tau \\ -v\tau \end{pmatrix}.$$

We thus find

$$a(v)T = \tau$$

$$c(v)T = -v\tau$$

or

$$\frac{c(v)}{a(v)} = -v.$$

Hence, we have

$$\Lambda(v) = \begin{pmatrix} a(v) & b(v) \\ -va(v) & d(v) \end{pmatrix}.$$

The inverse transformation must be given by $\Lambda(-v)$. Imposing the existence of an inverse gives

$$\begin{aligned}\begin{pmatrix} a(v) & b(v) \\ -va(v) & d(v) \end{pmatrix}\begin{pmatrix} a(-v) & b(-v) \\ va(-v) & d(-v) \end{pmatrix} &= \begin{pmatrix} a(v)a(-v)+vb(v)a(-v) & a(v)b(-v)+b(v)d(-v) \\ -va(v)a(-v)+vd(v)a(-v) & -va(v)b(-v)+d(v)d(-v) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

The 21 component of this matrix equation gives

$$a(v) = d(v). \tag{1}$$

The 11 component gives

$$a(v)a(-v) + vb(v)a(-v) = 1. \tag{2}$$

Now, consider rotation. In one dimension, one can rotate discretely by making $x \to -x$. Hence, the transformation from the primed frame to unprimed frame (which we just finished accomplishing by boosting by $-v$) is the same as rotating in the primed frame, boosting by *positive* $v$ and then rotating back. In equations, this set of transformations can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a(v) & b(v) \\ -va(v) & a(v) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a(v) & -b(v) \\ va(v) & a(v) \end{pmatrix} = \begin{pmatrix} a(-v) & b(-v) \\ va(-v) & a(-v) \end{pmatrix}$$

where the right hand side is determined by the previous method of boosting using $-v$. This implies

$$b(v) = -b(-v)$$

$$a(v) = a(-v)$$

Hence, our constraint equations are

$$a(v) = -vb(v) + \frac{1}{a(v)}.$$

This implies

$$b(v) = \frac{1 - a^2(v)}{va(v)}$$

To recap, imposing linear coordinate transformation for relativity implied a transformation matrix

$$\Lambda = \begin{pmatrix} a(v) & \frac{1-a^2(v)}{va(v)} \\ -va(v) & a(v) \end{pmatrix}.$$

Note that this answer is thus far consistent with Newtonian/Galilean relativity. If we take $a(v) = 1$, we find

$$\Lambda = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

which is just the Newtonian transformation which we found earlier. However, we will find $a(v) \neq 1$ except at $v = 0$ (at $v = 0$, we have $a(v) = 1$ by definition).

What Einstein's special relativity imposes is one extra condition that the speed of light is constant in all frames of reference. In particular, if in the unprimed coordinate system, the light signal is measured at coordinate $(T, T)$, in the primed coordinate system, the light signal will be measured at $(T', T')$ where

$$\begin{pmatrix} a(v) & \frac{1-a^2(v)}{va(v)} \\ -va(v) & a(v) \end{pmatrix}\begin{pmatrix} T \\ T \end{pmatrix} = \begin{pmatrix} T' \\ T' \end{pmatrix}.$$

Writing out the equations, we see

$$a(v) + \frac{1 - a^2(v)}{va(v)} = -va(v) + a(v)$$

or

$$a = \frac{1}{\sqrt{1-v^2}}$$

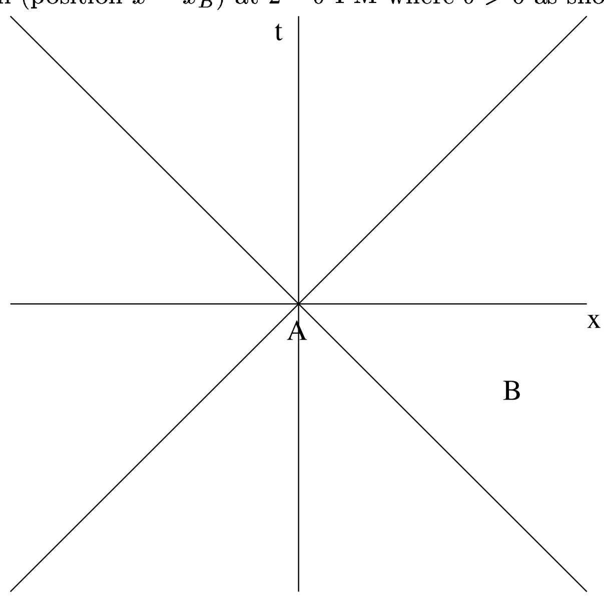
where we have chosen the sign such that $a = 1$ when $v = 0$.

Hence, let's recap:

1. Consider most general linear coordinate transformation between $(t, x)$ and $(t', x')$.
2. Considered the definition of relative velocity of the primed frame. i.e. the object at origin of the unprimed frame seen from the primed frame point of view.
3. Used the fact that we can go from the primed frame to the unprimed frame by boosting by $-v$ (inertial frames are equivalent).
4. We finally used the fact that boosting backwards from the primed frame to unprimed frame can be accomplished by a boost of $v$ if we rotate. (No preferred direction.)

**Exercise**

Suppose event A corresponds to someone stealing my cookie 2 PM at position $x = 0$. Suppose even B corresponds to Superman entering the physics lecture hall (position $x = x_B$) at $2 - \delta$ PM where $\delta > 0$ as shown in the figure below.




**a)** What is the time of event B according to the person moving with speed $v$ whose position coincides with the position of event A?

**b)** Can B event occur later than A event? Is this possible in the Newtonian/Galilean world?

**c)** Could Superman be responsible for my cookie being stolen (event A)?

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In 4-D, we can obtain any boost by rotating and boosting. For example, suppose you want to boost in the $\vec{v} = v_x\hat{x} + v_y\hat{y}$. First, rotate such that $v_y \to 0$, which means use the rotation matrix

$$R = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

where the $0-0$ entry corresponds to the time-time component and $\theta$ is chosen appropriately. Explicitly, we have

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \\ v_x \\ v_y \\ 0 \end{pmatrix} = \begin{pmatrix} \\ v'_x \\ 0 \\ 0 \end{pmatrix}$$

giving

$$\tan\theta = \frac{v_y}{v_X}$$
$$v'_x = \sqrt{v_x^2 + v_y^2}$$

The boost matrix will be

$$R^T \begin{pmatrix} \gamma' & -\gamma' v'_x & 0 & 0 \\ -\gamma' v'_x & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R = \begin{pmatrix} \gamma' & -\gamma' v_x & -\gamma' v_y & 0 \\ -\gamma' v_x & 1 + \frac{(\gamma'-1)v_x^2}{v_x'^2} & \frac{(\gamma'-1)v_x v_y}{v_x'^2} & 0 \\ -\gamma' v_y & \frac{(\gamma'-1)v_x v_y}{v_x'^2} & 1 + \frac{(\gamma'-1)v_y^2}{v_x'^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, one can compute the most general boost as

$$\Lambda(\vec{v}) = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + \frac{(\gamma-1)v_x^2}{v^2} & \frac{(\gamma-1)v_x v_y}{v^2} & \frac{(\gamma-1)v_x v_z}{v^2} \\ -\gamma v_y & \frac{(\gamma-1)v_x v_y}{v^2} & 1 + \frac{(\gamma-1)v_y^2}{v^2} & \frac{(\gamma-1)v_y v_z}{v^2} \\ -\gamma v_z & \frac{(\gamma-1)v_x v_z}{v^2} & \frac{(\gamma-1)v_y v_z}{v^2} & 1 + \frac{(\gamma-1)v_z^2}{v^2} \end{pmatrix}$$

where $\gamma \equiv 1/\sqrt{1-\vec{v}^2} = 1/\sqrt{1-(v_x^2+v_y^2+v_z^2)}$ where we have previously labeled $\gamma$ as $\gamma'$.