Lec 9: General Relativistic Tensors

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exercise

We learned in the last lecture that $g_{\mu\nu}$ acts like a potential. Does it have a natural geometrical meaning? Why?

1 Where are we and where are we going?

before:

a) Maxwell's eqs. defeats Newton's gravity $\longrightarrow$ special relativity

b) need inertial frame $\rightarrow$ equivalence principle $\longrightarrow$ new gravitational equation for a particle governed by $g_{\mu\nu}$.

c) What determines $g_{\mu\nu}$? reconcile with Newton's gravity $\longrightarrow$ matter density at least "perturbatively."

next:

a) What kind of "non-perturbative" gravitational equation governs $g_{\mu\nu}$? general relativistic tensor equations

b) What tensorial quantity involving derivatives of $g_{\mu\nu}$ (such that Newton's law can be recovered) are there? Riemman tensor and its contractions

c) What can I make of these tensors that gives Newton's law back in the weak field limit? matter density of Newton's gravity $\longrightarrow$stress energy tensor $\longrightarrow$ Einstein's equations

Einstein's equations governs many popularized phenomena including big bang cosmology and black holes. Imagine how all this followed from mere constant speed of light.

2 What is a general relativistic tensor?

Tensors were useful in special relativity because tensorial equations did not depend on which particular inertial frame they were defined. Indeed, at the very core of the special relativity is the philosophy there is no intrinsically preferred inertial frame of reference, which means that any fundamental physics equation should not depend on which inertial frame one is in. Tensors also allowed us to easily compute observables in different inertial frames if we knew what the quantities were in any one inertial frame. If we replace words "inertial frame" in the previous sentences with the word "coordinate" and replace the word "special" with "general", we have the motivation for looking for $g_{\mu\nu}$governing equations in terms of general relativitic tensors.

In the practical manner that we have been defining quantities thus far, general relativistic tensor is a mathematical quantity labeled by upper and lower indices which transform like the coordinate differentials with the upper and lower indices. First, let's see how coordinate differentials transform with upper and lower indices.

$$dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha \tag{1}$$

$$dx_\gamma \equiv g_{\gamma\lambda} dx^\lambda$$

Since $g_{\gamma\lambda} dx^\gamma dx^\lambda = dx_\lambda dx^\lambda$remains invariant under general coordinate transformations (as we learned in our exercise in the last lecture), we must have

$$dx_\gamma \to \frac{\partial x^\lambda}{\partial x'^\gamma} dx_\lambda.$$

exercise (From pg. 62 of your book) A Euclidean plane can be covered by Cartesian coordinates (x, y) or polar coordinates (r, θ). Write out explicitly the components of Eq. (1) where $dx'^\mu = (dr, d\theta)$.

A general relativistic tensor has the following transformation propery under *general coordinate transformations*:

$$dx^\mu \to \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha$$

$$T^{\alpha\beta\gamma...}{}_{\mu\nu\eta...} \to \frac{\partial x'^\alpha}{\partial x^{\alpha_1}} \frac{\partial x'^\beta}{\partial x^{\beta_1}} \frac{\partial x'^\gamma}{\partial x^{\gamma_1}} ... \frac{\partial x^{\mu_1}}{\partial x'^\mu} \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\eta_1}}{\partial x'^\eta} ... T^{\alpha_1\beta_1\gamma_1...}{}_{\mu_1\nu_1\eta_1...}$$

Note the similarity with how we defined Lorentz transformation tensors. What we have defined here are sometimes also called general coordinate transformation tensors. Also, note that strictly speaking, what we have been dealing with has been components of tensors and not tensors themselves. However, in this introductory course, we will not make such distinctions because for elementary computations, such distinctions are not useful.