

Solutions to Problem set 1

1st February 2006

Problems

1. Give the units of force, acceleration, and energy in geometricized units. In each case, what is the quantity that one must multiply to return to conventional units?

answer

As we learned in lecture 1, in geometricized units, we have

$$\text{force: } [F] = \frac{[M][L]}{[T]^2} \times \frac{1}{\frac{[L]^2}{[T]^2}} \times \frac{[L]}{[M]} = \text{dimensionless}$$

$$\text{acceleration: } [a] = \frac{[F]}{[M]} \times \frac{1}{[L]/[M]} = [L]^{-1}$$

$$\text{energy: } [E] = [F][L] = [L]$$

To get back to normal units, multiply the geometricized unit quantities by following:

$$\text{force} \rightarrow \text{force} \frac{c^4}{G_N}$$

$$\text{acceleration} \rightarrow \text{acceleration} c^2$$

$$\text{energy} \rightarrow \text{energy} \frac{c^4}{G_N}$$

2. Suppose Newton's second law is changed to

$$\vec{F} = m\ddot{\vec{x}} + \kappa\dot{\vec{x}}$$

where κ is a constant while the Newtonian gravitational force law remains the same. Suppose the force \vec{F} on mass m at point \vec{x} is generated gravitationally by a mass M at position \vec{X} .

- a) Just as in class, show that the system is invariant under the transformation

$$t \rightarrow t + \tau$$

$$\vec{x} \rightarrow R\vec{x}$$

where R is a rotation matrix satisfying $R^T R = \text{identity matrix}$ and τ is a constant.

answer

Gravitational law is

$$mM \frac{(\vec{X} - \vec{x})}{|\vec{X} - \vec{x}|^3} = m\ddot{\vec{x}} + \kappa\dot{\vec{x}}$$

Under the transformation

$$t \rightarrow t + \tau$$

$$\vec{x} \rightarrow R\vec{x}$$

we have

$$\begin{aligned} |\vec{X} - \vec{x}| &\rightarrow |R(\vec{X} - \vec{x})| = \sqrt{(\vec{X} - \vec{x})^T R^T R (\vec{X} - \vec{x})} = \sqrt{(\vec{X} - \vec{x})^T (\vec{X} - \vec{x})} \\ &= |\vec{X} - \vec{x}| \\ mM \frac{(\vec{X} - \vec{x})}{|\vec{X} - \vec{x}|^3} &= m\ddot{\vec{x}} + \kappa\dot{\vec{x}} \rightarrow mM \frac{R(\vec{X} - \vec{x})}{|\vec{X} - \vec{x}|^3} = mR\ddot{\vec{x}} + \kappa R\dot{\vec{x}} \end{aligned}$$

Hence, this modified Newton's law remains invariant under the rotation and time translation parts of the Galilean transformations.

b) Which part of the Galilean transformation is not obeyed by this modified second law?

answer

Clearly $\vec{x} \rightarrow \vec{x} + \vec{v}t$ does not leave $\kappa\dot{\vec{x}}$ invariant.

3. We know that $\epsilon^{ijk} \partial_j A_k = (\vec{\nabla} \times \vec{A})^i$ in Cartesian coordinates in Euclidean space.

a) Prove the identity

$$\epsilon_{ijk} \epsilon_{lm}{}^k = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

b) Use the index notation to show that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

answer

$$\textcircled{3} a) X \equiv \sum_{ijk} \epsilon_{emb} \overset{\text{term 1}}{\epsilon_{ij1} \epsilon_{em1}} + \overset{\text{term 2}}{\epsilon_{ij2} \epsilon_{em2}} + \overset{\text{term 3}}{\epsilon_{ij3} \epsilon_{em3}}$$

Property 1

Note X vanishes unless $i \neq j$ and $l \neq m$.

Property 2: For term 1, $\{i, j\} \in \{2, 3\} \Rightarrow \{l, m\}$
For term 2, $\{i, j\} \in \{1, 2\} \Rightarrow \{l, m\}$; etc.

2 cases: 1) Suppose $i=l$:

Then, unless $j=m$, X vanishes because of Property 1 and 2.

This case contributes $\delta_{ie} \delta_{jm}$

2) Suppose $i \neq l$:

Then, unless $j \neq m$, X vanishes because of Property 1 and 2. Furthermore, because of property 2, X vanishes unless $i=m$ and $j=l$.

This case contributes $-\delta_{im} \delta_{je}$ since $\epsilon_{ij1} \epsilon_{ji1}$ (no summation) = -1.

$$\therefore \epsilon_{ijk} \epsilon_{emb} = \delta_{ie} \delta_{jm} - \delta_{im} \delta_{je}$$

$$\begin{aligned} b) \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \epsilon_{ijk} \partial_j \epsilon_{ken} \partial_n A_m \\ &= (\delta_{ie} \delta_{jm} - \delta_{im} \delta_{je}) \partial_j \partial_n A_m \\ &= \partial_i \partial_m A_m - \partial_j^2 A_i \\ &= \boxed{\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}} \end{aligned}$$

4. Using the explicit form of the rotation matrix R given in lecture 1, verify that $(R^T)^i{}_j R^j{}_k = \delta^i{}_k$.

answer

$$R = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R^T = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, note

$$\begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly for others. Hence

$$R^T R = \text{unit matrix}$$

5. Derive

$$\phi(t, \vec{r}_1) = \frac{q}{4\pi} \frac{1}{r_* - \vec{v} \cdot \vec{r}_*}$$

from

$$\phi(t, \vec{r}_1) = \frac{1}{4\pi} \int d^3 r_2 \frac{\rho(t - r_{12}, \vec{r}_2)}{r_{12}}$$

where the notation is identical to that in the lecture notes when we talk about a point charge q with constant speed v moving along the x -axis with the initial condition of charge being at $x = 0$ at $t = 0$.

answer

Start with

$$\rho(t - r_{12}, \vec{r}_2) = q \delta^{(3)}(\vec{r}_2 - \hat{x}v(t - r_{12}))$$

$$r_{12} \equiv \sqrt{(r_1 - r_2)^i (r_1 - r_2)_i}$$

Hence,

$$\begin{aligned} \phi(t, \vec{r}_1) &= \frac{1}{4\pi} \int d^3 r_2 \frac{\rho(t - r_{12}, \vec{r}_2)}{r_{12}} \\ &= \frac{q}{4\pi} \int d^3 r_2 \frac{\delta(r_{2x} - v(t - \sqrt{(r_1 - r_2)^i (r_1 - r_2)_i})) \delta(r_{2y}) \delta(r_{2z})}{\sqrt{(r_1 - r_2)^i (r_1 - r_2)_i}} \\ &= \frac{q}{4\pi} \int dr_{2x} \frac{\delta(r_{2x} - v(t - \sqrt{(r_{1x} - r_{2x})^2 + r_{1y}^2 + r_{1z}^2}))}{\sqrt{(r_{1x} - r_{2x})^2 + r_{1y}^2 + r_{1z}^2}} \\ &= \frac{q}{4\pi} \int dr_{2x} \frac{\delta(r_{2x} - r_{2x*})}{(1 - \frac{v(r_{1x} - r_{2x})}{\sqrt{(r_{1x} - r_{2x})^2 + r_{1y}^2 + r_{1z}^2}}) \sqrt{(r_{1x} - r_{2x})^2 + r_{1y}^2 + r_{1z}^2}} \end{aligned}$$

where r_{2x*} satisfies $r_{2x*} - v(t - \sqrt{(r_{1x} - r_{2x*})^2 + r_{1y}^2 + r_{1z}^2}) = 0$. Writing $r_{1x} = x, r_{1y} = y$, and $r_{1z} = z$, we find

$$\phi(t, \vec{x}) = \frac{q}{4\pi} \frac{1}{(\sqrt{(x - r_{2x*})^2 + y^2 + z^2} - v(x - r_{2x*}))}$$

$$r_{2x*} - v(t - \sqrt{(x - r_{2x*})^2 + y^2 + z^2}) = 0$$

(1)

Let

$$r_* \equiv \sqrt{(x - r_{2x*})^2 + y^2 + z^2} \quad (2)$$

which has the interpretation of the distance between the observer and the charge at the time when the Coulomb field was “emitted.” We can then rewrite Eq. (1) as

$$r_{2x*} - v(t - r_*) = 0. \quad (3)$$

Let

$$t' \equiv t - r_*$$

which has the interpretation of retarded time (which is the time at which the Coulomb field was “emitted”) Eq. (3) then becomes

$$r_{2x*} = vt'$$

which in turn allows us to write Eq. (2) as

$$r_* = \sqrt{(x - vt')^2 + y^2 + z^2} \quad (4)$$

which is what we wanted. Finally, we can reexpress the potential as

$$\begin{aligned} \phi(t, \vec{x}) &= \frac{q}{4\pi} \frac{1}{(r_* - v(x - r_{2x*}))} \\ &= \frac{q}{4\pi} \frac{1}{r_* (1 - v \frac{(x - vt')}{r_*})}. \end{aligned}$$

In light of Eq. (4), let \vec{r}_* be the vector

$$\vec{r}_* \equiv (x - vt', y, z).$$

Since $x - vt'$ is just the \hat{x} component and $\vec{v} = v\hat{x}$, we arrive at

$$\phi(t, \vec{x}) = \frac{q}{4\pi} \frac{1}{r_* - \vec{v} \cdot \vec{r}_*}.$$