# PROBLEM SET 2

**due::** Thursday, February 9, 2006, in my mailbox

Problems

**1.:**

**a):** Within the framework of Newtonian gravity, write down a differential equation involving the mass density $j(\vec{x})$ governing the Newtonian gravitational potential $\phi$. Assume $j(\vec{x})$has a finite volume and spatial extension in which it is nonvanishing (a test object with a negligibly small mass $m$ has a potential energy $m\phi(\vec{x})$ at the location $\vec{x}$.)

**answer:**

In analogy with Coulomb's law in electromagnetism, we can immediately write down

$$\nabla^2\phi = 4\pi j(\vec{x}).$$

This would lead immediately to

$$\phi(\vec{x}) = -\int \frac{d^3x' j(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

which we are familiar from Newton's laws.

**b):** Calculate the Newtonian potential $\phi(\vec{x})$ for the mass density

$$j(\vec{x}) = M\delta^{(3)}(\vec{x}).$$

**answer:**

If for example, we have a point charge at the origin, we have

$$j(\vec{x}') = M\delta^{(3)}(\vec{x}')$$

which would give rise to the potential

$$\phi = \frac{-M}{|\vec{x}|}$$

which would give rise to the test charge potential energy of

$$m\phi = \frac{-Mm}{|\vec{x}|}$$

at point $\vec{x}$.

**2.:** Problem 2.3 on page 36.

**answer:**

a) If we have $x' = \gamma(x-vt)$, $y'=y$, $z'=z$, $t'=\gamma(t-vx)$, we have $x=\gamma(x'+vt')$, $y=y'$, $z=z'$, $t=\gamma(t'+vx')$ from the fact that the inverse of the boost matrix must take $v\to -v$. Now, using chain rule, we have

$$\begin{aligned}\partial_{t'} &= \frac{\partial t}{\partial t'}\partial_t + \frac{\partial x}{\partial t'}\partial_x \\ &= \gamma(\partial_t + v\partial_x)\end{aligned}$$

which we see transform "oppositely" to $t'$. Similar reasoning holds for $x'$ as well.

b) Since we have

$$x'^\mu = \Lambda^\mu{}_\alpha(v)x^\alpha$$

while

$$\begin{aligned}\frac{\partial x'^\mu}{\partial x'^\nu} = \delta^\mu{}_\nu &= \frac{\partial}{\partial x'^\nu}\Lambda^\mu{}_\alpha(v)x^\alpha \\ &= M^\gamma{}_\nu \frac{\partial}{\partial x^\gamma}\Lambda^\mu{}_\alpha(v)x^\alpha \\ &= M^\gamma{}_\nu\Lambda^\mu{}_\alpha(v)\frac{\partial}{\partial x^\gamma}x^\alpha \\ &= M^\gamma{}_\nu\Lambda^\mu{}_\alpha(v)\delta^\alpha{}_\gamma \\ &= M^\gamma{}_\nu\Lambda^\mu{}_\gamma(v)\end{aligned}$$

Hence, we see that $M$ must be the inverse matrix of $\Lambda$ which is what we wanted to show.

**3.:** We showed in lecture that the wave equation for light

$$\partial_k^2 E_i - \partial_t^2 E_i = 0$$

is not invariant under the transformation

$$(t, \vec{x}) \to (t, \vec{x} - \vec{v}t)$$

with $\vec{v}$ being a constant velocity. Show that it is invariant under the transformation $(t, \vec{x}) \to (t', \vec{x}')$ if

$$\begin{aligned} t' &= \gamma t - \gamma v x \\ x' &= \gamma x - \gamma v t \\ y' &= y \\ z' &= z \end{aligned}$$

**answer:**

By chain rule, we have

$$\begin{aligned} \partial_t &= \partial_t t' \partial_{t'} + \partial_t x' \partial_{x'} \\ &= \gamma \partial_{t'} - \gamma v \partial_{x'} \end{aligned}$$

$$\begin{aligned} \partial_x &= \partial_x t' \partial_{t'} + \partial_x x' \partial_{x'} \\ &= -\gamma v \partial_{t'} + \gamma \partial_{x'} \end{aligned}$$

while all other derivatives remain invariant. Hence, we find

$$\partial_t^2 = \gamma^2 (\partial_{t'}^2 - 2v \partial_{x'} \partial_{t'} + v^2 \partial_{x'}^2)$$

$$\partial_x^2 = \gamma^2 (\partial_{x'}^2 - 2v \partial_{x'} \partial_{t'} + v^2 \partial_{t'}^2)$$

Therefore, the part of the wave operator

$$\begin{aligned} \partial_x^2 - \partial_t^2 &= \gamma^2 [(1 - v^2) \partial_{x'}^2 - (1 - v^2) \partial_{t'}^2] \\ &= \partial_{x'}^2 - \partial_{t'}^2 \end{aligned}$$

Hence, the wave equation remains invariant under Lorentz transformations.

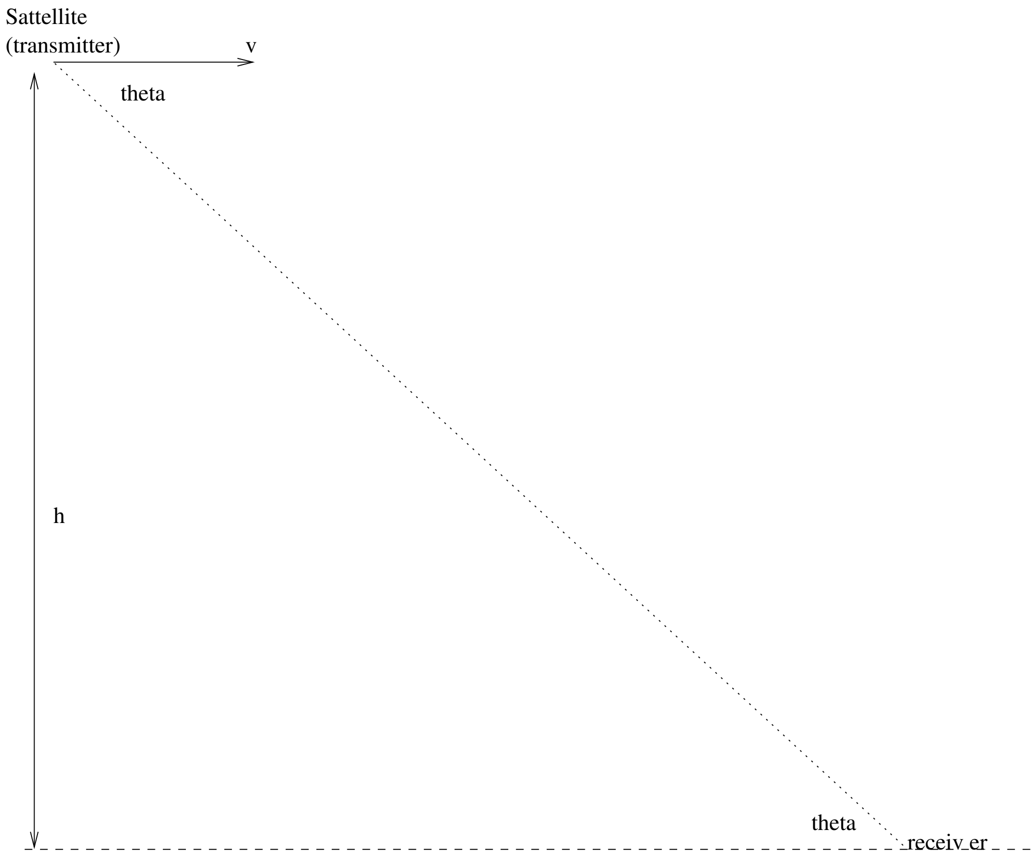
**4.:** Problem 2.6 on page 36.
**answer:**

Eqs. $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t - vx)$ give

$$\begin{aligned} x'^2 - t'^2 &= \gamma^2 (x^2 + v^2 t^2 - 2xvt - t^2 - v^2 x^2 + 2tvx) \\ &= \gamma^2 [x^2 (1 - v^2) - t^2 (1 - v^2)] \\ &= x^2 - t^2 \end{aligned}$$

Hence, the interval $s^2 = x^2 + y^2 + z^2 - t^2$ remains invariant under Lorentz transformations.

**5.:** Suppose a sattellite is moving across the sky at a height $h$ with velocity $v$ parallel to the ground such that the receiving antenna on the ground makes an angle $\theta$ as shown:




Here we are obviously assuming that the Earth is approximately flat for the distances of interest.

**a):** If the emission frequency from the sattellite rest frame is $\nu'$, compute the reception frequency $\nu_r$ on the ground. (Assume the first wave crest is emitted when the angle in the figure is $\theta$.)

**answer::**

We will label the $x$ axis parallel to the ground and $y$ axis the distance above the ground. Suppose the first wave crest is sent out at time $(t = 0, x = 0, y = h)$. From geometry, the receiver is then located at $(x = h/\tan\theta, y = 0)$. Since the receiver is then a distance $h/\sin\theta$ away, this wave crest arrives at the receiver $(t = h/\sin\theta, x = h/\tan\theta, y = 0)$. The second crest is sent out at a time $(t = t_1, x = vt_1, y = h)$. The distance from the emitter to the receiver is

$$d_2 \equiv \sqrt{h^2 + (\frac{h}{\tan\theta} - vt_1)^2}.$$

Hence, this second wave crest arrives at the receiver at the coordinate $(t = d_2 + t_1, x = h/\tan\theta, y = 0)$. The difference between the receiving time is

$$\frac{1}{\nu_r} = d_2 + t_1 - \frac{h}{\sin\theta}. \tag{0.1}$$

Now, $t_1$ is related to $\frac{1}{\nu'}$ by Lorentz transformations. By applying Lorentz transformations as usual, we see

$$t_1' = \gamma t_1 - \gamma v(vt_1) = \frac{t_1}{\gamma}.$$

Furthermore, since the first emission point $(t = 0, x = 0, y = h)$ remains unchanged by Lorentz transformations, we find

$$\nu' = \frac{\gamma}{t_1}.$$

Hence, we find from Eq. (0.1)

$$\frac{1}{\nu_r} = \sqrt{h^2 + (\frac{h}{\tan\theta} - v\frac{\gamma}{\nu'})^2} + \frac{\gamma}{\nu'} - \frac{h}{\sin\theta}.$$

**b):** Take the answer of result in part a) in the limit that $h \to \infty$.

**answer:**

First expand the answer from part a) as

$$\begin{aligned}\frac{1}{\nu_r} &= h\sqrt{1+(\frac{1}{\tan\theta}-\frac{v}{h}\frac{\gamma}{\nu'})^2}+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}\\ &= h\sqrt{1+\frac{1}{\tan^2\theta}+(\frac{v}{h}\frac{\gamma}{\nu'})^2-\frac{2\gamma v}{h\nu'\tan\theta}}+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}\\ &= h\sqrt{\frac{1}{\sin^2\theta}+(\frac{v}{h}\frac{\gamma}{\nu'})^2-\frac{2\gamma v}{h\nu'\tan\theta}}+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}\\ &= \frac{h}{\sin\theta}\sqrt{1+(\frac{v}{h}\frac{\gamma}{\nu'})^2\sin^2\theta-\frac{2\gamma v}{h\nu'}\cos\theta\sin\theta}\;+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}\end{aligned}$$

Expanding the squareroot to leading order in $1/h$, we find

$$\begin{aligned}\frac{1}{\nu_r} &= \frac{h}{\sin\theta}(1-\frac{\gamma v}{h\nu'}\cos\theta\sin\theta)\;+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}+O(\frac{1}{h^2})\\ &= -\frac{\gamma v}{\nu'}\cos\theta\;+\frac{\gamma}{\nu'}+O(\frac{1}{h^2})\\ &= \frac{\gamma}{\nu'}(1-v\cos\theta).\end{aligned}$$

Hence, we find

$$\nu_r=\frac{\nu'}{\gamma(1-v\cos\theta)}.$$

**c):** Evaluate the expression obtained for $\nu_r$ in part b) for $\theta=\pi/2$. Does the answer make sense?

**answer::**

Putting in $\theta=\pi/2$, we find

$$\nu_r=\frac{\nu'}{\gamma}.$$

This makes sense since when the motion of the emitter is perpendicular to the signal being received, only time dilation can effect the signal due to the fact that spatial dimensions perpendicular to the direction of motion is not transformed by Lorentz transformations.

**d):** Expand the answer to part c) (i.e. the expression for $\nu_r$ for $\theta=\pi/2$) to leading nonvanishing term in $v$, as $v\to 0$. (i.e. Expand to leading term containing $v$ that does not vanish as $v\to 0$. For example, for $f(v)=1+5v+2v^2+3v^3$, the expansion of $f(v)$ to leading nonvanishing $v$ dependent term is $1+5v$.) Is the leading nonvanishing $v$ dependent term linear in $v$ or quadratic in $v$? How does this leading nonvanishing $v$ dependence compare with the Doppler shift we derived for 1-dimensional motion that we derived previously? (Previously, we derived $\nu_r=\nu'\sqrt{\frac{1+v}{1-v}}$.)

**answer::**

Since $\gamma=\frac{1}{\sqrt{1-v^2}}$ is a function of $v^2$, the expansion as $v\to 0$ is the same as the expansion of $1/\sqrt{1-x}$ as $x\to 0$. Hence, we have

$$\nu_r=\nu'(1-\frac{1}{2}v^2)$$

Hence, the leading $v$ dependent term is quadratic in $v$. Expanding the 1-dimensional Doppler effect to leading order $v$, we find

$$\begin{aligned}\nu_r &= \nu'\sqrt{\frac{1+v}{1-v}}\\ &= \nu'(1+2v)\end{aligned}$$

Hence, the 1-dimensional Doppler effect has a leading linear dependence on $v$ with a plus sign while the $\theta=\pi/2$ case with for our higher spatial dimensional case has a quadratic dependence on $v$ with a minus sign.

**e):** Take the result of part *a*) and take the limit $\theta=0$.

**answer:**

From the intermediate steps of part b), we first write

$$\frac{1}{\nu_r}=\frac{h}{\sin\theta}\sqrt{1+(\frac{v}{h}\frac{\gamma}{\nu'})^2\sin^2\theta-\frac{2\gamma v}{h\nu'}\cos\theta\sin\theta}\;+\frac{\gamma}{\nu'}-\frac{h}{\sin\theta}.$$

In the limit $\theta\to 0$, we find

$$\begin{aligned}\frac{1}{\nu_r} &= -\frac{\gamma v}{\nu'}+\frac{\gamma}{\nu'}\\ &= \frac{\gamma}{\nu'}(1-v).\end{aligned}$$

which is what we have for the one dimensional Doppler shift.

**6.:** While moving at a high speed $v$, particle X of rest mass 135 MeV decays symmetrically into two massless particles (decaying means that there is no X particle left over at the end; only two massless particles are in the final state of the system). The final state energy of each massless particle is 100 MeV.

**a):** Find speed $v$ of particle X.

**answer:**

By energy conservation

$$2E_f = M\gamma$$

where $E_f = 100\text{MeV}$ is the energy of the final state massless particles and $M = 135\text{MeV}$. Therefore, we have

$$\begin{aligned} v &= \sqrt{1-(\frac{M}{2E_f})^2} \\ &= \frac{\sqrt{871}}{40} \approx 0.74 \end{aligned}$$

**b):** Find the angle $\theta$ (in the laboratory system) between the spatial momenta of the massless particles.

**answer:**

By squaring the 4-momenta, we find

$$\begin{aligned} M^2 &= (\tilde{p}_3 + \tilde{p}_4)^2 \\ &= 2\tilde{p}_3 \cdot \tilde{p}_4 \\ &= 2E_f^2(1-\cos\theta) \end{aligned}$$

where $\tilde{p}_3$ and $\tilde{p}_4$ are 4-momenta of the final state massless particles. Hence, the angle is given by

$$\begin{aligned} \cos\theta &= 1 - \frac{M^2}{2E_f^2} \\ &= \frac{71}{800} \end{aligned}$$

or

$$\theta = 1.48$$

**c):** Given that 1 eV (electronvolt) $= 1.6 \times 10^{-19}$J, what is the rest mass of particle X in units of kilograms?

**answer:**

By definition of rest mass, we have

$$Mc^2 = 135\text{MeV}\frac{10^6\text{eV}}{1\text{MeV}}\frac{1.6\times10^{-19}\text{J}}{1\text{eV}}$$

Since $c = 3 \times 10^8\text{m/s}$, we find

$$M = 2.4 \times 10^{-28}\text{kg}$$

which is about 1/7 of the mass of a proton.