PROBLEM SET 3

due:: Thursday, March 2, 2006

Problems

1.: Consider the Maxwell tensor $F_{\alpha\beta}$.

a): Compute $F_{\alpha\beta}F^{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$.

b): How does your answer in part a) transform under Lorentz transformations?

2.: Use the Jacobian formalism in multivariable calculus to show that the integration measure d^4x is invariant under Lorentz transformations for which $\det\Lambda = 1$. (Hint: Use the fact that $\eta_{\alpha\beta}$ is invariant under Lorentz transformations and the general matrix property $\det[M_1 M_2] = \det M_1 \det M_2$ to show that $\det\Lambda^\mu{}_\nu = \pm 1$; then choose $\det\Lambda^\mu{}_\nu = 1$ to show that d^4x is invariant under Lorentz transformations that is continuously connected to the identity transformation.)

3.: Suppose we define

$$j^\alpha = \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \frac{d\vec{x}_n^\alpha(t)}{dt}$$

which has the natural interperation of $(\rho, \vec{j})$ that we used in class. In this exercise, we would like to prove that j^α is a special relativistic 4-vector.

a): One starts by writing

$$\begin{aligned} j^\alpha &= \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \frac{dx_n^\alpha(t)}{dt} \\ &= \int dt' \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t'))\delta(t - t') \frac{dx_n^\alpha(t')}{dt'} \\ &= \int dt' \sum_n e_n \delta^{(4)}(x^\beta - x_n^\beta(t')) \frac{dx_n^\alpha(t')}{dt'} \end{aligned}$$

where we have defined $x_n^0 = t'$ and $x^0 = t$. (Note that x^β inside the delta function is a shorthand for denoting a 4-vector and does not refer to any specific β component.) Now, the proper time of the nth particle is given by

$$d\tau_n = dt' \sqrt{1 - (\frac{d\vec{x}_n}{dt'})^2}.$$

Use this to rewrite j^α as

$$j^\alpha = \sum_n \int d\tau_n e_n \delta^{(4)}(x^\beta - x_n^\beta(t'(\tau_n))) \frac{dx_n^\alpha}{d\tau_n}.$$

b): Now justify

$$\delta^{(4)}(x^\gamma - x_n^\gamma)$$

is invariant under Lorentz transformations. (Hint: Use change of variables for integration measure to express $\delta^{(4)}(\Lambda^\gamma{}_\beta[x^\beta - x_n^\beta])$ in terms of $\delta^{(4)}(x^\gamma - x_n^\gamma)$ and $\det\Lambda$.)

c): Explain why we can therefore conclude that j^α is a 4-vector.

4.: On the surface of a 2-sphere (surface of a ball with unit radius), the Euclidean metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Compute the non-vanishing components of the Christoffel symbol for this metric.