Physics 721 Final Exam
Spring, 2005

Helpful formulae:

Legendre Polynomials

$$P_0(x) = 1,\;\; P_1(x) = x,\;\; P_2(x) = \frac{1}{2}(3x^2 - 1),\;\; P_3(x) = \frac{1}{2}(5x^3 - 3x),\;\; P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Directions

Most of the credit will be given for the functional dependence of variables. You have 2 hours to complete the exam.

1. (20 pt) Starting directly from Maxwell's equations, derive the attenuation length of a linear isotropic material having a good conductivity of σ.

answer

We know that a wave of the form

$$\vec{E} = \hat{\epsilon} e^{i\vec{k}\cdot\vec{x} - iwt}$$

where $\hat{\epsilon} \cdot \vec{k} = 0$ (from $\vec{\nabla} \cdot \vec{E} = 0$) are solutions to Maxwell eqs. The inverse of the imaginary part of $\vec{k}$ gives the attenuation length since the imaginary part gives rise to exponential damping. Using

$$\vec{J} + \partial_t \vec{E}\epsilon = \vec{\nabla} \times \vec{B}/\mu, \qquad -\partial_t \vec{B} = \vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{J} = \sigma \vec{E}$$

we would write

$$\vec{B} = \frac{1}{w}\vec{k} \times \vec{E}, \qquad (\sigma - iw\epsilon)\vec{E} = -i\frac{k^2}{w}\vec{E}/\mu.$$

Hence, we conclude

$$k \approx \sqrt{\mu w \sigma}(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$$

The attenuation length is therefore

$$\delta \approx \sqrt{\frac{2}{w\sigma\mu}}$$

which vanishes in the limit $\sigma \to \infty$.

2. (20 pt) A thin spherical shell of charge with radius R is centered at the origin. Let θ denote the angle measured from the $z-$axis. The sphere carries a surface charge density given by $\sigma = \sigma_0 \cos^2\theta$ where σ_0 is a constant. Find the potential everywhere.

answer

By dimensional analysis and the fact that an axially symmetric potential Φ must remain regular at $r = 0$ and $r = \infty$, we can write

$$\Phi = \begin{cases} \sum_{l=0}^{\infty} A_l (\frac{r}{R})^l \frac{1}{R} P_l(\cos\theta) & r < R \\ \sum_{l=0}^{\infty} B_l (\frac{R}{r})^{l+1} \frac{1}{R} P_l(\cos\theta) & r \geq R \end{cases}$$

Since the potential must be continuous at $r = R$, we have

$$A_l = B_l.$$

Furthermore, the pill box boundary condition for the electric field above and below the surface of the sphere is

$$-\partial_r \Phi|_{r=R^+} - (-\partial_r \Phi|_{r=R^-}) = \sigma_0 \cos^2\theta.$$

Using the fact that

$$\cos^2\theta = \frac{1}{3}(2P_2(\cos\theta) + P_0(\cos\theta)),$$

one finds

$$A_0 = B_0 = \frac{\sigma_0 R^2}{3}$$

$$A_2 = B_2 = \frac{2\sigma_0 R^2}{15}$$

and all other coefficients 0.

3. (20 pt) A long straight wire carrying current I is placed a distance a above a semi-infinite magnetic medium of permeability μ. Calculate the force per unit length acting on the wire (specify the magnitude and direction).

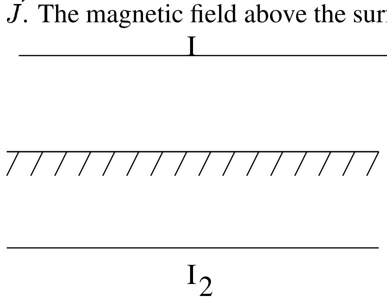
answer

The boundary condition at the surface is

$$H_{||z=0^+} = H_{||z=0^-}$$

$$B_{\perp z=0^+} = B_{\perp z=0^-}$$

where $\vec{H} \equiv \vec{B}/\mu$ below the surface and the perpendicular and parallel subscripts denote the orientation with respect to the plane of the surface. We know that the B-field strength due to a straight wire current is proportional to the current and $1/(2\pi r)$ by $\nabla \times \vec{B} = \vec{J}$. The magnetic field above the surface can be computed by introducing an image current



I_2 as shown in the figure: If we choose the wire axis to be along the $\hat{y}$ and the direction perpendicular to the surface to be $\hat{z}$ (pointing from I_2 to I_1), the parallel component of the magnetic field at $(x, y, z) = (0, y, 0^+)$ is

$$B_{||}|_{z=0^+}\hat{x} = k\hat{x}\frac{1}{a}(I_2 - I)$$

where k is a numerical factor. Similarly, for below the surface ($z = 0^-$), we choose an image current I_3 to write

$$B_{||}|_{z=0^-}\hat{x} = -k\hat{x}\frac{1}{a}I_3.$$

Hence the boundary condition involving μ gives

$$I_2 - I = -I_3/\mu. \qquad (1)$$

The remaining boundary condition requires us to project the perpendicular component. This is easily accomplished by noting that $r = a/\cos\theta$ where θ is the azimuthal angle about the wire axis and noting that the perpendicular component is proportional to $\sin\theta$. Unlike the parallel component, perpendicular component from I and its image add when the image current is moving in the same direction:

$$\sin\theta\frac{I}{(a/\cos\theta)} + \sin\theta\frac{I_2}{(a/\cos\theta)} = \sin\theta\frac{I_3}{(a/\cos\theta)}. \tag{2}$$

Combining Eqs. (1) and (2), one finds

$$I_2 = I\frac{(\mu-1)}{(\mu+1)}.$$

Since the force per unit length on wire 1 is proportional to BI and the direction will be down for moving in the same direction, we find

$$\text{force per unit length} = \frac{-\hat{z}I^2}{4\pi a}\frac{\mu-1}{\mu+1}.$$

As expected, it vanishes when $\mu = 1$ and the "mirror" becomes perfect as $\mu \to \infty$.

4. (20 pt)

a) Suppose A^μ is the relativistic vector potential. Write the 4 Maxwell equations in terms of it.

answer

The inhomgeneous equations are given by

$$\partial_\mu F^{\mu\nu} = J^\nu$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The homogeneous equations are given by

$$\partial_\mu \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = 0$$

which is by construction an identity.

b) Solve the differential equation

$$(\partial_t^2 - \nabla^2)A^\mu(t,x) = Q^\mu \frac{1}{r_0^2}\delta(t)\delta(r-r_0)\delta(\theta-\pi/2)\delta(\phi)$$

for $A^\mu(t,x)$ where $r_0 > 0$ and Q^μ is a constant Lorentz 4-vector and $\{r,\theta,\phi\}$ are spherical radial coordinates. Assume that the boundary condition for $A^\mu(t,x)$ is that it vanishes at $t = -\infty$.

answer

Recognizing that the right hand side is a 4 dimensional delta function, one can immediately write down the answer as just the 4D Green's function

$$A^\mu = \frac{Q^\mu}{2\pi}\delta((x^0)^2 - (x-r_0)^2 - y^2 - z^2)\theta(x^0)$$

which certainly satisfies the boundary condition. This Green's function can be guessed from thinking about causality.

c) Suppose one is given that $Q^\mu = (0,1,0,0)$ in the Cartesian frame of part a) which we call frame 1. What is the value of A^μ in frame 2 which is moving with respect to frame 1 with speed v in the $\hat{x}$ direction?

answer

Since A^μ is a 4-vector, it tranforms to

$$A^\mu \quad \rightarrow \quad \Lambda^\mu{}_\nu A^\nu(x) = \frac{Q'^\mu}{2\pi}\delta(t^2 - (x - r_0)^2 - y^2 - z^2)\theta(t)$$

where

$$Q^\mu = (-\gamma v, \gamma, 0, 0)$$

If we express (t, x, y, z) in terms of the new coordinates, it is

$$\theta(t) = \theta(t')$$

$$\begin{aligned} t^2 - (x - r_0)^2 - y^2 - z^2 &= (t' - t'_0)^2 - (x' - x'_0)^2 - y'^2 - z'^2 \\ &= (t' + \gamma v r_0)^2 - (x' - \gamma r_0)^2 - y'^2 - z'^2 \end{aligned}$$

5. (20 pt) You know by now that the radiation field due to an accelerated charge is given by

$$\vec{E}(t, \vec{x}) = \frac{q}{4\pi}[\frac{\hat{n} - \vec{v}}{\gamma^2(1 - \vec{v}\cdot\hat{n})^3 R^2}]_{ret} + \frac{q}{4\pi}[\frac{\hat{n} \times [(\hat{n} - \vec{v}) \times \dot{\vec{v}}]}{(1 - \vec{v}\cdot\hat{n})^3 R}]|_{ret}$$

where $R \equiv |\vec{x} - \vec{r}(t)|$, $\vec{v} \equiv \frac{d}{dt}\vec{r}$ is the velocity of the accelerated charge, and $\hat{n} = \hat{R}$. Derive in the **nonrelativistic limit** the leading contribution to the polarized differential scattering cross section

$$\frac{d\sigma}{d\Omega}(\hat{\epsilon}_f, \hat{n}_f; \hat{\epsilon}_i, \hat{n}_i)$$

from a charged particle constrained to move only along the $\hat{x}$ direction where the incident light porlarization is $\hat{\epsilon}_i = \hat{x}$ and propagation direction is $\hat{n}_i = \hat{y}$, while the final state light polarization is $\hat{\epsilon}_f$ and propagation direction is $\hat{n}_f = \hat{z}$. Assume that the charge of the scatterer is q and the mass of the scatterer is m.

answer

Intuitively, power is proportional to electric field squared. Hence, we write

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\epsilon}_f, \hat{n}_f; \hat{\epsilon}_i, \hat{n}_i) &= \frac{|\vec{E}_{sc} \cdot \hat{\epsilon}_f^*|^2 R^2}{|\vec{E}_{inc}|^2} \\ &\approx (\frac{q}{4\pi})^2 \frac{|\hat{\epsilon}_f^* \cdot (\hat{n}_f \times [\hat{n}_f \times \dot{\vec{v}}])|^2}{|\vec{E}_{inc}|^2} \end{aligned}$$

where $\vec{E}_{sc}$ corresponds to scattered electric field amplitude and $\vec{E}_{inc}$ corresponds to the incident electric field amplitude. Note we have kept the leading order term in $\vec{v}$ (nonrelativistic limit). Since we are given $\hat{n}_f = \hat{z}$ and $\vec{v}$ is constrained to be along $\hat{x}$ direction,

$$\begin{aligned} \hat{n}_f \times [\hat{n}_f \times \dot{\vec{v}}] &= \hat{z} \times [\hat{z} \times \hat{x}]\dot{v} \\ &= -\hat{x}\dot{v}. \end{aligned}$$

Finally, note that because the charge is constrained to move only along the $\hat{x}$ direction, $m\dot{v} = q|\vec{E}_{inc}|$. Hence we arrive at

$$\frac{d\sigma}{d\Omega}(\hat{\epsilon}_f, \hat{n}_f; \hat{\epsilon}_i, \hat{n}_i) = (\frac{q^2}{4\pi})^2 \frac{|\hat{\epsilon}_f^* \cdot \hat{x}|^2}{m^2}$$

where one can choose any $\hat{\epsilon}_f$ that satisfies $\hat{\epsilon}_f \cdot \hat{n}_f = 0$.