

Homework 3 (due: 10/18/07) solutions:

1. Dodelson pg. 136, #5

The time-space component of the Einstein tensor is given by

$$G^0_i = R^0_i = g^{00}R_{0i} = \frac{1}{-(1+2\Psi)}R_{0i} \quad (1)$$

since the metric is still diagonal. Explicitly

$$R_{0i} = \Gamma_{0i,\alpha}^\alpha - \Gamma_{0\alpha,i}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{0i}^\beta - \Gamma_{\beta i}^\alpha \Gamma_{0\alpha}^\beta.$$

Now, we know from our previous Einstein tensor computation experience that the R_{0i} vanishes if the perturbations are 0. Hence, we only need to compute the perturbation part:

$$\begin{aligned} \Gamma_{0i,\alpha}^\alpha &= \left(\frac{1}{2}g^{\alpha\lambda}[g_{0\lambda,i} + g_{i\lambda,0} - g_{0i,\lambda}]\right)_{,\alpha} \\ &= \left(\frac{1}{2}[-2\Psi_{,i}g^{\alpha 0} + g^{\alpha\lambda}g_{i\lambda,0}]\right)_{,\alpha} \\ &= \frac{1}{2}[(-2\Psi_{,i}g^{00})_{,0} + (g^{0\lambda}g_{i\lambda,0})_{,0} + (g^{k\lambda}g_{i\lambda,0})_{,k}] \\ &= \frac{1}{2}[2\Psi_{,i0} + (g^{ii}g_{ii,0})_{,i}] \quad (\text{no sum}) \\ &= \frac{1}{2}[2\Psi_{,i0} + \left(\frac{1}{a^2(1+2\Phi)}(a^2(1+2\Phi))_{,0}\right)_{,i}] \\ &= \Psi_{,i0} + \Phi_{,0i} \end{aligned}$$

Similarly, one can compute the rest as

$$\begin{aligned} -\Gamma_{0\alpha,i}^\alpha &= -3\Phi_{,0i} - \Psi_{,0i} \\ \Gamma_{\beta\alpha}^\alpha \Gamma_{0i}^\beta &= \frac{\partial_t a}{a}[3\Phi_{,i} + 4\Psi_{,i}] \\ -\Gamma_{\beta i}^\alpha \Gamma_{0\alpha}^\beta &= \frac{-\partial_t a}{a}[3\Phi_{,i} + 2\Psi_{,i}] \end{aligned}$$

Hence, we find

$$R_{0i} = -2\Phi_{,0i} + \frac{\partial_t a}{a}2\Psi_{,i}$$

Using Eq. (1), we thus find

$$G^0_i = 2\partial_i \left[\frac{\dot{\Phi}}{a} - H\Psi \right]$$

where the dot corresponds to conformal time derivative. In Fourier space, $\partial_i \rightarrow ik_i$. Hence, we find

$$G^0_i = 2ik_i \left[\frac{\dot{\Phi}}{a} - H\Psi \right].$$

Since the stress energy tensor is

$$T^0_i = a \int \frac{d^3p}{(2\pi)^3} p_i [g_{dm} f_{dm}(t, \vec{x}, \vec{p}) + g_b f_b(t, \vec{x}, \vec{p}) + g_\gamma f_\gamma(t, \vec{x}, \vec{p}) + g_\nu f_\nu(t, \vec{x}, \vec{p})],$$

if we recognize

$$\begin{aligned} f_\gamma &= f_\gamma^{(0)} - p \frac{\partial f_\gamma^{(0)}}{\partial p} \Theta \\ f_\nu &= f_\nu^{(0)} - p \frac{\partial f_\nu^{(0)}}{\partial p} \mathcal{N} \end{aligned}$$

and use the nonrelativistic approximation for the dark matter and baryons, we find

$$\begin{aligned} T^0_i &= a[\rho_{dm}v_i + \rho_b(v_b)_i + \int \frac{d^3p}{(2\pi)^3} p_i (-g_\gamma p \frac{\partial f_\gamma^{(0)}}{\partial p} \Theta - g_\nu p \frac{\partial f_\nu^{(0)}}{\partial p} \mathcal{N})] \\ &= a[\rho_{dm}v_i + \rho_b(v_b)_i + 4 \int \frac{d^3p}{(2\pi)^3} p_i (g_\gamma f_\gamma^{(0)} \Theta + g_\nu f_\nu^{(0)} \mathcal{N})] \end{aligned}$$

where we have used $p_i = p\hat{p}_i$ and integrated by parts. Now, Fourier transforming and using no-vorticity condition:

$$\begin{aligned} v_i &\rightarrow v\hat{k}_i \\ (v_b)_i &\rightarrow v_b\hat{k}_i. \end{aligned}$$

Hence, we have

$$2ik_i[\frac{\dot{\Phi}}{a} - H\Psi] = 8\pi Ga[\rho_{dm}v\hat{k}_i + \rho_bv_b\hat{k}_i + 4 \int \frac{d^3p}{(2\pi)^3} p_i (g_\gamma f_\gamma^{(0)} \Theta + g_\nu f_\nu^{(0)} \mathcal{N})].$$

Dotting both sides by \hat{k}^i (note that $\hat{k}^i = \hat{k}_i$), we have

$$\begin{aligned} \frac{\dot{\Phi}}{a} - H\Psi &= \frac{4\pi}{ik} Ga[\rho_{dm}v + \rho_bv_b + 4 \int \frac{d^3p}{(2\pi)^3} p\mu (g_\gamma f_\gamma^{(0)} \Theta(t, \mu) + g_\nu f_\nu^{(0)} \mathcal{N}(t, \mu))] \\ &= \frac{4\pi}{ik} Ga[\rho_{dm}v + \rho_bv_b + 4 \int \frac{dpp^2 4\pi}{(2\pi)^3} p (g_\gamma f_\gamma^{(0)} \int_{-1}^1 \frac{d\mu}{2} \mu \Theta + g_\nu f_\nu^{(0)} \int_{-1}^1 \frac{d\mu}{2} \mu \mathcal{N})] \\ &= \frac{4\pi}{ik} Ga[\rho_{dm}v + \rho_bv_b + (4 \int \frac{d^3p}{(2\pi)^3} p g_\gamma f_\gamma^{(0)}) \int_{-1}^1 \frac{d\mu}{2} \mu \Theta + (4 \int \frac{d^3p}{(2\pi)^3} p g_\nu f_\nu^{(0)}) \int_{-1}^1 \frac{d\mu}{2} \mu \mathcal{N}] \end{aligned}$$

Using the definition

$$\Theta_1 \equiv \frac{1}{(-i)} \int_{-1}^1 \frac{d\mu}{2} P_1(\mu) \Theta,$$

we find

$$\dot{\Phi} - aH\Psi = \frac{4\pi}{ik} Ga^2[\rho_{dm}v + \rho_bv_b - 4i\rho_\gamma\Theta_1 - 4i\rho_\nu\mathcal{N}_1] \quad (2)$$

where we note that we have been assuming both photons and neutrinos are massless.

2. Dodelson pg. 137, #6

The time-time Einstein equation is

$$k^2\Phi + 3\frac{\dot{a}}{a}(\dot{\Phi} - \Psi\frac{\dot{a}}{a}) = 4\pi Ga^2[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0].$$

Since $\dot{\Phi} - \Psi\frac{\dot{a}}{a} = \dot{\Phi} - aH\Psi$ appears in Eq. (2), we can combine the two equations to give

$$k^2\Phi + \frac{\dot{a}}{a} \frac{12\pi}{ik} Ga^2[\rho_{dm}v + \rho_bv_b - 4i\rho_\gamma\Theta_1 - 4i\rho_\nu\mathcal{N}_1] = 4\pi Ga^2[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0]$$

which implies

$$k^2\Phi + \frac{\dot{a}}{a} \frac{12\pi}{ik} Ga^2[\rho_m v_m - 4i\rho_r\Theta_{r,1}] = 4\pi Ga^2[\rho_m\delta_m + 4\rho_r\Theta_{r,0}]$$

or equivalently

$$k^2\Phi = 4\pi Ga^2[\rho_m\delta_m + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}(4\rho_r\Theta_{r,1} + i\rho_mv_m)].$$

When the wavelength is much shorter than the Horizon, we recover Poisson's equation

$$\frac{k^2}{a^2}\Phi \approx 4\pi G[\rho_m\delta_m + 4\rho_r\Theta_{r,0}]$$

3. Dodelson pg. 213, #5

The wavenumber at equality can be defined as

$$k_{eq} = a_{eq}H(a_{eq}).$$

At matter radiation equality, $\rho_m = \rho_r$. Hence,

$$H(a_{eq}) = \sqrt{\frac{8\pi G}{3}2\rho_m(a_{eq})}.$$

Dividing by $\sqrt{\rho_c(t_0)} = \sqrt{\frac{3}{8\pi G}H_0}$ and setting $\rho_m(a_{eq}) = \rho_{m0}(a_0/a_{eq})^3$, we find

$$\frac{H(a_{eq})}{H_0} = \sqrt{2\Omega_{m0}\left(\frac{a_0}{a_{eq}}\right)^3}.$$

Hence, if we set $a_0 = 1$, we find

$$k_{eq} = \sqrt{\frac{2\Omega_m H_0^2}{a_{eq}}}$$

where we have dropped the subscript 0 on Ω_m following typical convention. Setting $a_{eq} = 4.15 \times 10^{-5}/(\Omega_m h^2)$ gives

$$\begin{aligned} k_{eq} &= \Omega_m h^2 100 \text{km/s/Mpc} / (3 \times 10^5 \text{km/s}) \sqrt{\frac{2}{4.15 \times 10^{-5}}} \\ &= 0.073 \text{Mpc}^{-1} \Omega_m h^2. \end{aligned}$$

Suppose one sets

$$k_{eq}^{(2)} \eta_{eq} = 1.$$

Since

$$\left(\frac{\dot{a}}{a^2}\right)^2 \approx \frac{8\pi}{3}G(\rho_{m0}\left(\frac{a_0}{a}\right)^3 + \rho_{\gamma 0}\left(\frac{a_0}{a}\right)^4)$$

we have

$$\eta \approx \int \frac{da}{\sqrt{\frac{8\pi}{3}G(\rho_{m0}a + \rho_{\gamma 0})}}$$

where we have set $a_0 = 1$. Since $\rho_{m0}a_{eq} \approx \rho_{\gamma 0}$,

$$\begin{aligned} \eta_{eq} &\approx \frac{1}{\sqrt{\frac{8\pi}{3}G\rho_{m0}}} \int_0^{a_{eq}} \frac{da}{\sqrt{a + a_{eq}}} \\ &= \frac{2(\sqrt{2} - 1)\sqrt{a_{eq}}}{\sqrt{\frac{8\pi}{3}G\rho_{m0}}} \\ &= \frac{2(\sqrt{2} - 1)\sqrt{a_{eq}}}{H_0\sqrt{\Omega_m}} = \frac{2(\sqrt{2} - 1)\sqrt{4.15 \times 10^{-5}}}{100 \text{km/s/Mpc} \Omega_m h^2}. \end{aligned}$$

Hence, one would find instead

$$k_{eq}^{(2)} \approx 0.062 \text{Mpc}^{-1} \Omega_m h^2$$

which is about 14% lower than k_{eq} (or k_{eq} is about 17% higher than $k_{eq}^{(2)}$).