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Renormalization group approach to satisfiability

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Abstract – Satisfiability is a classic problem in computational complexity theory, in which one wishes to determine whether an assignment of values to a collection of Boolean variables exists in which all of a collection of clauses composed of logical ORs of these variables is true. Here, a renormalization group transformation is constructed and used to relate the properties of satisfiability problems with different numbers of variables in each clause. The transformation yields new insight into phase transitions delineating "hard" and "easy" satisfiability problems.

Computational complexity theory addresses the question of how fast the resources required to solve a given problem grow with the size of the input needed to specify the problem [1]. P is the class of problems that can be solved in polynomial time, which means a time that grows as a polynomial of the size of the problem specification, while NP is the class of problems for which a solution can be verified in polynomial time. Whether or not P is distinct from NP has been a central unanswered question in computational complexity theory for decades [2].

Satisfiability (SAT) is a classic problem in computational complexity. An often-studied type of SAT is K-SAT, in which one attempts to find assignment of N variables such that the conjunction (AND) of M constraints, or clauses, each of which is the disjunction (OR) of K literals, each literal being either a negated or un-negated variable, is true. (This way of writing the problem, as a conjunction of clauses that are disjunctions, is called conjunctive normal form.) For example, the 3-SAT instance with the 4 variables x_1, x_2, x_3, and x_4 and the four clauses

$$\begin{aligned} &(x_1=1 \text{ OR } x_2=0 \text{ OR } x_4=1)\\ \text{AND}\quad &(x_1=0 \text{ OR } x_3=1 \text{ OR } x_4=0)\\ \text{AND}\quad &(x_2=1 \text{ OR } x_3=1 \text{ OR } x_4=1)\\ \text{AND}\quad &(x_1=0 \text{ OR } x_3=0 \text{ OR } x_4=1), \end{aligned} \tag{1}$$

is satisfiable because it it is true for the assignments $x_1=1$, $x_2=1$, $x_3=1$, $x_4=1$. Below, we will write satisfiability problems in conjunctive normal form using the notation of [3], where the ANDs and ORs are implied and the literals have positive or negative signs depending on whether or not they are negated. For example, the expression of eq. (1) is written

$$(1\ -2\ 4),\ (-1\ 3\ -4),\ (2\ 3\ 4),\ (-1\ -3\ 4).$$

2-SAT can be solved in polynomial time [1], while K-SAT with $K \geqslant 3$ is known to be NP-complete [4]: if a polynomial algorithm for 3-SAT exists, then P is equal to NP. The complexity of SAT is intimately related to the presence of phase transitions [5–11]. For random problems with N variables, M clauses, and K literals per clause, as M is increased there is a phase transition from a satisfiable phase, in which almost all random instances are satisfiable, to an unsatisfiable phase, in which almost all random instances are unsatisfiable. The most difficult instances are near this SAT-unSAT transition. It has also been shown that there is a transition as the parameter K is changed between 2 and 3, at $K_c \sim 2.4$, at which the nature of the SAT-unSAT transition changes [5].

Here, we investigate the relationship between satisfiability problems with different values of K by constructing a renormalization group transformation, similar to those used for phase transition problems [12–15], that reduces the number of degrees of freedom, while possibly increasing the number and range of interactions [16] (which in this context is the number of literals per clause). To do this, we note that the expression

$$((\mathcal{A}_1\ x),\ (\mathcal{A}_2\ x), \ldots (\mathcal{A}_P\ x),$$

$$(\mathcal{B}_1\ -x),\ (\mathcal{B}_2\ -x)) \ldots, (\mathcal{B}_Q\ -x))$$

is satisfiable if and only if

$$((\mathcal{A}_1\ \mathcal{B}_1),\ (\mathcal{A}_1\ \mathcal{B}_2), \ldots (\mathcal{A}_1\ \mathcal{B}_Q),$$
$$(\mathcal{A}_2\ \mathcal{B}_1),\ (\mathcal{A}_2\ \mathcal{B}_2), \ldots (\mathcal{A}_2\ \mathcal{B}_Q),$$
$$\ldots,$$
$$(\mathcal{A}_P\ \mathcal{B}_1),\ (\mathcal{A}_P\ \mathcal{B}_2),\ \ldots,\ (\mathcal{A}_{\mathcal{P}}\ \mathcal{B}_Q))$$

is satisfiable. Here, the $\mathcal{A}_i$'s and $\mathcal{B}_i$'s are arbitrary clauses and x is a variable. (The easiest way to see the equivalence is to note that both expressions are satisfiable if and only if ($\mathcal{A}_1$ AND $\mathcal{A}_2$ AND $\ldots\mathcal{A}_P$) OR ($\mathcal{B}_1$ AND $\mathcal{B}_2$ AND $\ldots\mathcal{B}_Q$) is.) The first step of the renormalization procedure is to use this identity to eliminate a given variable. In this step, P clauses in which a given variable comes in un-negated and Q clauses in which the same variable comes in negated are eliminated and replaced with PQ "resolution" [17] clauses. Thus, eliminating a "frustrated" [18] variable (one that enters into different clauses negated and un-negated) increases the number of clauses if PQ-(P+Q)>0. The resolution of two clauses of length K_i and K_j has length K_i+K_j-2. Note that resolving two 2-clauses yields a 2-clause, resolving a 2-clause with a clause of length $\mathcal{K}\geqslant 3$ yields a clause length $\mathcal{K}$, and resolving two clauses with lengths $\mathcal{K}_1\geqslant 3$ and $\mathcal{K}_2\geqslant 3$ yields a clause with length greater than both $\mathcal{K}_1$ and $\mathcal{K}_2$.

One then simplifies the resulting satisfiability expression by noting that

1) Duplicate clauses are redundant,

2) Duplicate literals in a given clause are redundant,

3) If a variable enters into one clause both negated and un-negated, then the clause must be true and can be removed,

4) If a clause has one literal, then the value of the corresponding variable is determined, and

5) If a subset of the literals in a clause comprise a different clause, then the clause with more literals is redundant.

This last point means, for example, that if an expression contains both (1 3 −4 5) and (1 3), then (1 3 −4 5) can be removed, because it is satisfied automatically if (1 3) is satisfied.

This procedure is known in computer science as "the Davis-Putnam procedure of 1960 [19] with subsumption [20]," and was originally proposed as a method for solving satisfiability instances. It does not perform well in practice [21], and has been proven to require exponential time on some instances [22,23]. However, here the aim is not to solve a given instance, but rather to investigate the "flow" of the problem itself as variables are eliminated [12,24]. In particular, this renormalization group (RG) transformation provides a natural framework for understanding a phase transitions between "easy" and "hard" satisfiability problems identified in [5].

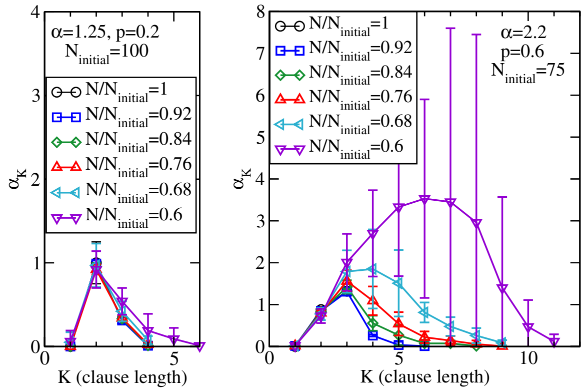

Fig. 1: Plot of α_K, the ratio of M_K, the number of clauses of length K, to N, the number of variables, at different stages of the renormalization process. The points plotted are the mean and standard deviation of the results from five independent system realizations. The parameters are chosen to be at the SAT-unSAT transition with $p=0.2<p_c$ (left panel) and $p=0.6>p_c$ (right panel). When $p>p_c$ the clause length increases markedly and the number of clauses grows enormously.

We present evidence that the change in the nature of the SAT-unSAT phase transition at the critical value $K_c\sim 2.4$ [5] is intimately related to whether or not the number of clauses proliferates exponentially upon repeated application of the renormalization group (RG) transformation. Note that when $K=2$ the clause length decreases upon renormalization, since the resolution of two 2-clauses is a 2-clause, so no clause gets longer, and some of the resulting clauses have a duplicate literal and so get shorter. Having a large number of 2-clauses limits the growth in the number of long clauses because of subsumption, so there is a qualitative difference in the behavior depending on whether the ratio of the number of 2-clauses to the number of variables grows or shrinks upon renormalization.

We show numerical data for an RG implementation in which successive variables are chosen randomly and eliminated if they occur in a clause of minimum length. This procedure is used because it focuses on short clauses, which when they are present tend to subsume the long clauses. Figure 1 shows α_K, the ratio of M_K, the number of clauses of length K to N, the number of variables remaining in the problem, as a function of K, as the RG proceeds. The average and standard deviation of numerical data from 5 realizations at the SAT-unSAT transition with $p=0.2$ and $p=0.6$, where $p=K-2$, are shown (using parameter values for the transition locations from [5]). Large numbers of long clauses are generated when $p=0.6>p_c$ and not when $p=0.2<p_c$.

It has been proven that the SAT-unSAT transition for 2-SAT occurs at $\alpha\equiv M/N=1$ [25], and for $2\leqslant K<2.4$, the SAT-unSAT transition is believed to occur when $\alpha_2=1$ [5]. Figure 2 (left) shows that when $K=2.2$,

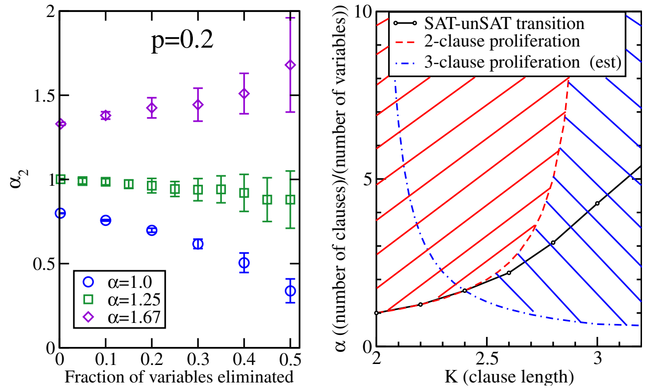

Fig. 2: Left: plot of α_2, the ratio of the number of 2-clauses to the number of variables, *vs.* N, the number of undecimated variables, for instances with $p = 0.2$ and different initial values of α. The numerical data are averages and standard deviations of five realizations of systems of size 500 when the initial $\alpha = 1$, size 400 when the initial $\alpha = \alpha_c = 1.2$, and size 300 when the initial $\alpha = 1.67$. The numerical data are consistent with the hypothesis that when $K < K_c$ the SAT-unSAT transition occurs when α_2 neither decreases nor increases upon renormalization. Right: schematic phase diagram showing the SAT-unSAT transition (using data of ref. [5]), the region in which the number of 2-clauses increases upon renormalization (the red (online) hatched region in the left of the figure) and an estimate of the region in which the number of clauses with $K \geqslant 3$ increases upon renormalization (the blue (online) hatched region in the right of the figure). The SAT-unSAT transition line crosses into the region in which long clauses proliferate exponentially at the intersection of the three lines. The estimate for 3-clause proliferation given in the text yields an intersection at ($K = 2.4$, $\alpha = 1.25$).

α_2 increases upon renormalization when $\alpha_2 > 1$ and decreases upon renormalization when $\alpha_2 < 1$. An approximate analytic expression for α_2, the ratio of the number of two-clauses to the number of variables, as a function of ρ, the fraction of variables eliminated, can be obtained by considering only cases in which a variable is in one two-clause negated and in any number of two-clauses un-negated, or vice versa. Within this approximation, eliminating a variable causes the number of variables to decrease by one, and also the number of clauses to decrease by one. Denoting the number of iterations by τ, the number of two-clauses after τ iterations as $M_2(\tau)$, and the ratio of the number of two-clauses to the number of variables at time τ as $\alpha_2(\tau) = M_2(\tau)/(N - \tau)$, one obtains

$$\alpha_2(\tau) = M_2(\tau)/(N-\tau) = (M_2(\tau-1)-1)/(N-\tau) = (M_2(0)-\tau)/(N-\tau), \quad (2)$$

so that as a function of $\rho = \tau/N$ (N is the number of variables before any decimations have been made), one has

$$\alpha_2(\rho) = (\alpha_2(0) - \rho)/(1-\rho). \quad (3)$$

In this expression, α_2 increases as ρ increases when $\alpha_2(0) > 1$, and decreases as ρ increases when $\alpha_2(0) < 1$ [26]. Because adding additional three-clauses does not affect the behavior of the two-clauses, and because two-clauses are much more restrictive than longer clauses, the two-clauses dominate the problem whenever $\alpha_2 > 1$. We estimate that a set of M_3 3-clauses would proliferate in the absence of 2-clauses when $3M_3/2 = N_{initial}$, the initial number of variables. This estimate, the analog of the result for two-clauses, follows from ignoring fluctuations and setting the number of literals in all 3-clauses to twice the number of variables, which means that on average each variable enters into one 3-clause negated and one 3-clause un-negated. Then, eliminating one variable on average yields two less literals and one less variable, so that the number of literals in all clauses of length greater than two remains equal to twice the number of variables. However, because proliferating 2-clauses subsume the 3-clauses, adding 3-clauses to the 2-clauses changes the nature of the SAT-unSAT transition only when enough 3-clauses have been added so that the SAT-unSAT transition occurs with $\alpha_2 < 1$. These results are consistent with those of Achlioptas *et al.* [27], who demonstrate that when $K < 2.4$ an assignment of variables that satisfies all the two-clauses also satisfies all the three-clauses with probability that approaches unity as $N \to \infty$.

In the regime in which $\alpha_2 < 1$, under renormalization the 2-clauses disappear and so the clauses all become longer. Since very long clauses are ORs of many literals and hence easy to satisfy, the number of clauses must go up sufficiently fast for the problem to be difficult to solve —at large K, the SAT-unSAT transition occurs when the ratio of the number of clauses to the number of variables is $\propto 2^K$ [11]. When $K > K_c$, near the SAT-unSAT transition we expect the number of clauses to grow geometrically with iteration number, and the numerical data are consistent with the maximum number of clauses obtained during the renormalization process increasing exponentially with the initial number of variables, with an exponent that increases monotonically with $(M/N)_{initial}$, the initial ratio of the number of clauses to the number of variables. Figure 2 (right) shows phase boundary lines for the SAT-unSAT transition, for the onset of increase in the number of 2-clauses, and our estimate for the onset of proliferation of clauses of length greater than or equal to three.

When $K > K_c$, the renormalization transformation causes both the typical clause length and the total number of clauses to grow. The value of α at which clauses proliferate is much lower than that at which standard algorithms take a long time on average, but may signify the onset of a region of "heavy tails" [28,29] in which standard algorithms run quickly on average but require exponential time on a small fraction of realizations. This type of behavior is very reminiscent of "Griffiths phases" [30] that occur in many disordered condensed matter systems. The SAT-UNSAT transition itself [5–10,31] occurs at a larger value of $\alpha = M/N$, which we conjecture is when the rate of exponential growth reaches a critical value.

It is interesting to consider the "replica-symmetry breaking" (RSB) transition [7,9,10,31] at which the space of satisfying assignments breaks up into many disconnected pieces in terms of the RG transformation. The RSB transition, which occurs at values of α greater than that at which clauses proliferate and smaller than that of the SAT-unSAT transition, is in the regime in which a given variable is in many clauses both un-negated and negated, so each decimation generates many new clauses that would not be generated if one of the possible assignments for the decimated variable was eliminated. Moreover, the value of a decimated variable is often implicitly determined by that of an undecimated one, say y, when a newly generated clause of the form $(\ldots y \ldots - y \ldots)$ is eliminated in the subsumption step (in other words, the compound clause is always true because for a given value of y one can fix the eliminated variable so that both original clauses are satisfied). At the very end of the RG process, when the number of variables is small and the newly generated clauses are short, in the replica-symmetry-broken phase one expects that fixing the value of one of the remaining variables will fix the values of all the eliminated variables.

Because of the exponential clause proliferation, numerical investigation of the RSB and SAT-unSAT transitions using this renormalization group is limited to small sizes. However, the renormalization group may still be useful for investigating these transitions using analytic techniques appropriate for large K [10,11] for any $K > K_c$, though it will be necessary to understand how to account for possible RG-induced correlations between clauses.

Finally, we note that when $K > K_c$, the number of clauses continues to increase under renormalization until it is no longer unlikely that a given compound clause contains a repeated variable (in addition to the decimated one), which we expect to occur when the renormalized clause length is of order $\sqrt{N}$ [32]. It appears that there is no impediment to the growth in the effective value of K until it is of order $\sqrt{N}$, where the problem specification itself is exponentially large in N.

In summary, a transformation inspired by the renormalization group is constructed and used to relate the behavior of satisfiability problems with different values of K, the number of literals per clause. The transformation provides useful insight into previously identified phase transitions of satisfiability problems.

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