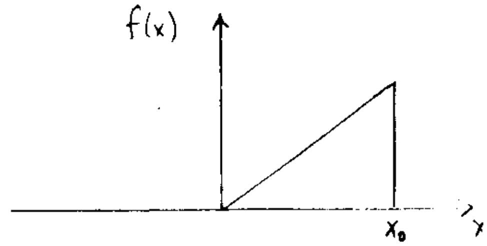
Homework 3 Solutions

W1) We are given

$$f(x) = \begin{cases} Ax & 0 \le x \le x_0 \\ 0 & \text{elsewhere} \end{cases}$$



(a) I will treat $f(x)$ as a probability distribution, so I want to normalize as follows

$$\int_{\text{all } x} f(x)dx = 1 = \int_0^{x_0} Ax\, dx = A\frac{x_0^2}{2} \Rightarrow \boxed{A = 2/x_0^2}$$

(b) $\bar{x} = \frac{1}{N}\int_{\text{all } x} x\, n(x)\, dx$. Using $f(x) = \frac{1}{N}n(x)$ we have

$$\bar{x} = \int_{\text{all } x} x f(x)\, dx = A\int_0^{x_0} x^2 dx = A\frac{x_0^3}{3} = \left(\frac{2}{x_0^2}\right)\left(\frac{x_0^3}{3}\right) \Rightarrow \boxed{\bar{x} = \frac{2}{3}x_0}$$

(c) To get x_{RMS} we first need $\overline{x^2}$:

$$\overline{x^2} = \int_{\text{all } x} x^2 f(x)\, dx = A\int_0^{x_0} x^3 dx = A\frac{x_0^4}{4} = \left(\frac{2}{x_0^2}\right)\left(\frac{x_0^4}{4}\right) = \frac{x_0^2}{2}$$

so

$$x_{RMS} = \sqrt{\overline{x^2}} \Rightarrow \boxed{x_{RMS} = \frac{1}{\sqrt{2}}x_0}$$

W2) (a) To find the fraction of atoms in each state we use the Maxwell-Boltzmann formula. Combining equations (10.4) and (10.3)

$E_3 = 0.2\text{eV}$
$E_2 = 0.1\text{eV}$
$E_1 = 0$

$$n_i = B g_i e^{-E_i/kT}$$

where B is a normalization constant and where the degeneracy factors g_i are all $g_i = 1$.

For $T = 300\text{K}$ we get

$$kT = (8.617 \times 10^{-5}\text{eV/K})\cdot(300\text{ K}) = 2.59 \times 10^{-2}\text{eV} = .0259\text{ eV}$$

If we have N atoms, then the constant B is determined as follows

$$N = n_1 + n_2 + n_3 = B\left[e^{-E_1/kT} + e^{-E_2/kT} + e^{-E_3/kT}\right] = B\left[1 + .0209 + .0004\right]$$

$$= 1.0213\ B \quad \Rightarrow \quad B = N/1.0213$$

The fraction of atoms in state i is

$$f_i = n_i/N = Bg_i e^{-E_i/kT}/N = \left(\frac{1}{1.0213}\right) e^{-E_i/kT}$$

Thus

$$f_1 = \left(\frac{1}{1.0213}\right) e^{0} \quad \Rightarrow \quad \boxed{f_1 = 0.979 = 97.9\%}$$

$$f_2 = \left(\frac{1}{1.0213}\right) e^{-0.1/.0259} \quad \Rightarrow \quad \boxed{f_2 = 0.0205 = 2.05\%}$$

$$f_3 = \left(\frac{1}{1.0213}\right) e^{-0.2/.0259} \quad \Rightarrow \quad \boxed{f_3 = 0.00043 = 0.043\%}$$

(b) $$\bar{E} = \frac{1}{N}\sum_i E_i n_i = \sum_i' E_i f_i$$

$$= (0.979)(0) + (0.0205)(0.1\,eV) + (0.00043)(0.2\,eV) \Rightarrow \boxed{\bar{E} = 2.13 \times 10^{-3}\ eV}$$

Notice that in this example $\bar{E}$ is small compared to kT. Even if the atom had more states at higher energies, $\bar{E}$ would not be affected much since these states would have very small f's.

10-6 **(a)** $n_1 + n_2 \approx 10^{20}$

$$\frac{n_2}{n_1} = \exp\left(\frac{-4.86\ \text{eV} \times 1.602 \times 10^{-19}\ \text{J/eV}}{1.38 \times 10^{-23}\ \text{J/K} \times 1.600 \times 10^{3}\ \text{K}}\right) = 4.98 \times 10^{-16}.$$

Assuming $n_1 \approx 10^{20}$,

$$n_2 \approx n_1\left(4.98 \times 10^{-16}\right) = \left(10^{20}\right)\left(4.98 \times 10^{-16}\right) = 4.98 \times 10^{4}.$$

(b) Power emitted $=$ number of photons emitted/s $\times$ (energy/photon)

$$= \left(\frac{1}{\tau}\right) \times n_2 \times 4.86\ \text{eV}$$

$$= 10^{7}\ \text{s}^{-1} \times 4.88 \times 10^{4} \times 4.86\ \text{eV} \times 1.602 \times 10^{-19}\ \text{J/eV}$$

$$= 3.88 \times 10^{-7}\ \text{J/s}$$

$$= 0.388\ \mu\text{W}$$

10-8 (a) Replacing E with K, the kinetic energy, we have $n(K)dK = \left[\frac{2\pi(N/V)}{(\pi k_B T)^{3/2}}\right] K^{1/2} e^{-K/k_B T} dK$.

The most probable value of kinetic energy, K_{mp} occurs where $K^{1/2} e^{-K/k_B T}$ has a maximum: $\frac{d\left[K^{1/2} e^{-K/k_B T}\right]}{dK} = \left(\frac{1}{2}\right) K^{-1/2} e^{-K/k_B T} + K^{1/2} e^{-K/k_B T}\left(\frac{-1}{k_B T}\right) = 0$ or

$\left(e^{-K/k_B T}\right)\left[\frac{1}{2K^{1/2}} - \frac{K^{1/2}}{k_B T}\right] = 0$ $e^{-K/k_B T} = 0$ implies $K = \infty$, a minimum. $\left[\frac{1}{2K^{1/2}} - \frac{K^{1/2}}{k_B T}\right] = 0$ implies $K_{mp} = \frac{k_B T}{2}$.

(b)

$$\overline{K} = \frac{1}{N/V}\int_{K=0}^{\infty} K n(K) dK = \frac{2\pi}{(\pi k_B T)^{3/2}}\int_{K=0}^{\infty} K^{3/2} e^{-K/k_B T} dK = \left[\frac{2\pi}{(\pi k_B T)^{3/2}}\right]\left[\frac{6(\pi k_B T)^{1/2}}{(8)(1/k_B T)^2}\right]$$

$$= \frac{(3/2)(\pi k_B T)^{1/2}(k_B T)^2}{(\pi)^{1/2}(k_B T)^{3/2}} = \frac{3}{2} k_B T$$

Note that our result $\overline{K} = \frac{3}{2} k_B T$ agrees with the equipartition theorem.

(c)

$$K_{rms} = \left[\frac{1}{N/V}\int_0^{\infty} K^2 n(K) dK\right]^{1/2} = \left[\frac{2\pi}{(\pi k_B T)^{3/2}}\int_0^{\infty} K^{5/2} n(K) dK\right]^{1/2}$$

$$= \left[\frac{2\pi}{(\pi k_B T)^{3/2}} \frac{(30)(\pi k_B T)^{1/2}}{\left[(16)(1/k_B T)^3\right]}\right]^{1/2} = \left[\frac{(30/8)(k_B T)^3}{k_B T}\right]^{1/2} = \left[\frac{15}{4}(k_B T)^2\right]^{1/2}$$

or $K_{rms} = \left(\frac{15}{4}\right)^{1/2} k_B T$.

3-2 Assume that your skin can be considered a blackbody. One can then use Wien's displacement law, $\lambda_{max} T = 0.2898 \times 10^{-2}\ \text{m}\cdot\text{K}$ with $T = 35\ ^0\text{C} = 308\ \text{K}$ to find

$$\lambda_{max} = \frac{0.2898 \times 10^{-2}\ \text{m}\cdot\text{K}}{308\ \text{K}} = 9.41 \times 10^{-6}\ \text{m} = 9\,410\ \text{nm}.$$

3-4 (a) From Stefan's law, one has $\frac{P}{A} = \sigma T^4$. Therefore,

$$\frac{P}{A} = (5.7 \times 10^{-8}\ \text{W/m}^2 K^4)(3000\ \text{K})^4 = 4.62 \times 10^6\ \text{W/m}^2.$$

(b)

$$A = \frac{P}{4.62 \times 10^6\ \text{W/m}^2} = \frac{75\ \text{W}}{4.62 \times 10^6\ \text{W/m}^2} = 16.2\ \text{mm}^2.$$

W3) (a) We need to know the surface area of the sun. For a sphere

$$A = 4\pi R^2 = 4\pi (7\times10^8 m) = 6.16\times10^{18} m^2$$

The total radiated power will be

$$P = (\sigma T^4)\cdot A = (5.67\times10^{-8} W/m^2\cdot K^4)(5800 K)^4 (6.16\times10^{18} m^2)$$

$$\boxed{P = 3.59\times10^{26} W}$$

(b) We have $P = \frac{dE}{dt} = 3.59\times10^{26}$ Joules/sec. If $E = mc^2$, then

$$\frac{dE}{dt} = c^2 \frac{dm}{dt} \Rightarrow \frac{dm}{dt} = \frac{dE}{dt}/c^2 = (3.59\times10^{26} J/s)/(3\times10^8 m/s)^2 \Rightarrow$$

$$\boxed{\frac{dm}{dt} = 4.39\times10^9 kg/s}$$

(c) During the lifetime of the sun, we expect the mass to decrease by a total of about

$$\Delta m = 0.01 m = 2\times10^{28} kg$$

So the lifetime is about

$$\Delta t = (\Delta m)/\frac{dm}{dt} = (2\times10^{28} kg)/(4.39\times10^9 kg/s) = 4.56\times10^{18} s$$

$$\boxed{\Delta t = 4.56\times10^{18} s = 1.44\times10^{11} \text{ years}}$$