Homework 7 Solutions

6-6 $$\psi(x) = A\cos kx + B\sin kx$$

$$\frac{\partial \psi}{\partial x} = -kA\sin kx + kB\cos kx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 A\cos kx - k^2 B\sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E-U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E-U)\psi$ or

$$-k^2(A\cos kx + B\sin kx) = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx).$$

Therefore $E = \frac{\hbar^2 k^2}{2m}$.

6-10 $$E_n = \frac{n^2 h^2}{8mL^2}$$

$$\frac{h^2}{8mL^2} = \frac{\left(6.63\times 10^{-34}\text{ Js}\right)^2}{8\left(9.11\times 10^{-31}\text{ kg}\right)\left(10^{-10}\text{ m}\right)^2} = 6.03\times 10^{-18}\text{ J} = 37.7\text{ eV}$$

(a) $E_1 = 37.7$ eV
$E_2 = 37.7\times 2^2 = 151$ eV
$E_3 = 37.7\times 3^2 = 339$ eV
$E_4 = 37.7\times 4^2 = 603$ eV

(b) $$hf = \frac{hc}{\lambda} = E_{n_i} - E_{n_f}$$

$$\lambda = \frac{hc}{E_{n_i} - E_{n_f}} = \frac{1\,240\text{ eV}\cdot\text{nm}}{E_{n_i} - E_{n_f}}$$

For $n_i = 4$, $n_f = 1$, $E_{n_i} - E_{n_f} = 603\text{ eV} - 37.7\text{ eV} = 565\text{ eV}$, $\lambda = 2.19$ nm
$n_i = 4$, $n_f = 2$, $\lambda = 2.75$ nm
$n_i = 4$, $n_f = 3$, $\lambda = 4.70$ nm
$n_i = 3$, $n_f = 1$, $\lambda = 4.12$ nm
$n_i = 3$, $n_f = 2$, $\lambda = 6.59$ nm
$n_i = 2$, $n_f = 1$, $\lambda = 10.9$ nm

6-16 (a) $\psi(x) = A\sin\left(\frac{\pi x}{L}\right)$, $L = 3$ Å. Normalization requires

$$1 = \int_0^L |\psi|^2 dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{LA^2}{2}$$

so $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \int_0^{L/3} |\psi|^2 dx = \left(\frac{2}{L}\right)\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{\pi}\int_0^{\pi/3} \sin^2\phi\, d\phi = \frac{2}{\pi}\left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8}\right] = 0.195\,5.$$

(b) $\psi = A\sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

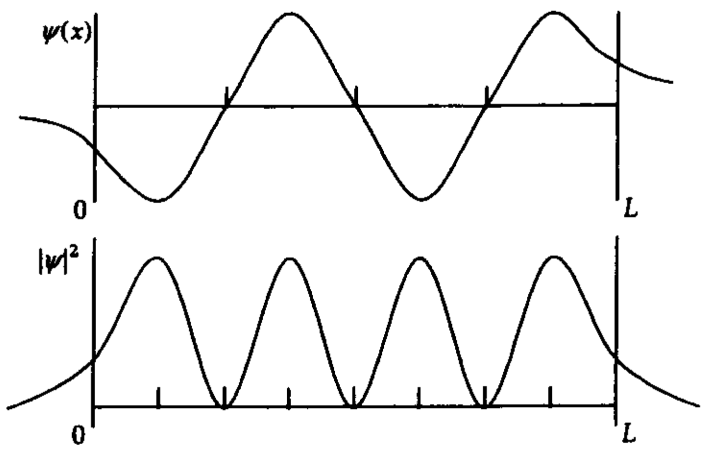
$$P = \frac{2}{L}\int_0^{L/3}\sin^2\left(\frac{100\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{100\pi}\right)\int_0^{100\pi/3}\sin^2\phi\, d\phi = \frac{1}{50\pi}\left[\frac{100\pi}{6} - \frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.331\,9$$

(c) Yes: For large quantum numbers the probability approaches $\frac{1}{3}$.

6-21 $n = 4$

$n = 4$



Note that the $n = 4$ wavefunction has three nodes and is antisymmetric about the midpoint of the well.

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$ with $U(x) = \frac{1}{2}m\omega^2x^2$ for the quantum oscillator. Differentiating $\psi(x) = Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx} = -2\alpha x\psi(x) + C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2} = -\frac{2\alpha x d\psi}{dx} - 2\alpha\psi(x) - (2\alpha x)Ce^{-\alpha x^2} = (2\alpha x)^2\psi(x) - 6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$. Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and $E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty}|\psi(x)|^2 dx = 2C^2\int x^2 e^{-2\alpha x^2}dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-29 (a) Normalization requires $1=\int_{-\infty}^{\infty}|\psi|^2dx=C^2\int_0^{\infty}e^{-2x}(1-e^{-x})^2dx=C^2\int_0^{\infty}(e^{-2x}-2e^{-3x}+e^{-4x})dx$.

The integrals are elementary and give $1=C^2\left\{\frac{1}{2}-2\left(\frac{1}{3}\right)+\frac{1}{4}\right\}=\frac{C^2}{12}$. The proper units for C are those of $(\text{length})^{-1/2}$ thus, normalization requires $C=(12)^{1/2}\ \text{nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where ψ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0=\frac{d\psi}{dx}=C\{-e^{-x}+2e^{-2x}\}=Ce^{-x}\{2e^{-x}-1\}.$$

The RHS vanishes when $x=\infty$ (a minimum), and when $2e^{-x}=1$, or $x=\ln 2\ \text{nm}$. Thus, the most likely position is at $x_p=\ln 2\ \text{nm}=0.693\ \text{nm}$.

(c) The average position is calculated from

$$\langle x\rangle=\int_{-\infty}^{\infty}x|\psi|^2dx=C^2\int_0^{\infty}xe^{-2x}(1-e^{-x})^2dx=C^2\int_0^{\infty}x(e^{-2x}-2e^{-3x}+e^{-4x})dx.$$

The integrals are readily evaluated with the help of the formula $\int_0^{\infty}xe^{-ax}dx=\frac{1}{a^2}$ to get $\langle x\rangle=C^2\left\{\frac{1}{4}-2\left(\frac{1}{9}\right)+\frac{1}{16}\right\}=C^2\left\{\frac{13}{144}\right\}$. Substituting $C^2=12\ \text{nm}^{-1}$ gives

$$\langle x\rangle=\frac{13}{12}\ \text{nm}=1.083\ \text{nm}.$$

We see that $\langle x\rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of x larger than x_p are weighted more heavily in the calculation of the average.

W10) The wave function for the state is $\psi(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi}{L} x$, and this applies in the region $0 < x < L$ with $\psi = 0$ outside that region.

(a) The probability distribution is symmetric about the center of the well, so we expect $\langle x \rangle = \frac{1}{2} L$. To verify that we have

$$\langle x \rangle = \int \psi^*(x)\, x\, \psi(x)\, dx = \left(\frac{2}{L}\right) \int_0^L x \sin^2 \frac{\pi}{L} x \, dx$$

This is integral # 399 in the web table

$$\langle x \rangle = \left(\frac{2}{L}\right) \left[\frac{x^2}{4} - \left(\frac{L}{\pi}\right) \frac{\sin 2\pi x/L}{4} - \left(\frac{L}{\pi}\right)^2 \frac{\cos 2\pi x/L}{8} \right]_0^L$$

$$= \left(\frac{2}{L}\right) \left[\frac{L^2}{4} - \frac{L}{\pi} \left(\frac{\sin 2\pi}{4} \overset{0}{} - \frac{\sin 0}{4} \overset{0}{} \right) - \left(\frac{L}{\pi}\right)^2 \left(\frac{\cos 2\pi}{8} - \frac{\cos 0}{8} \right) \right]$$

$$= \left(\frac{2}{L}\right)\left(\frac{L^2}{4}\right) \Rightarrow \boxed{\langle x \rangle = \frac{L}{2}} \quad \text{as expected}$$

(b) $\langle x^2 \rangle = \int \psi^*(x)\, x^2\, \psi(x)\, dx = \left(\frac{2}{L}\right) \int_0^L x^2 \sin^2 \frac{\pi}{L} x \, dx$ [see integral 400]

$$= \frac{2}{L} \left[\frac{x^3}{6} - \left(\frac{L}{\pi} \frac{x^2}{4} - \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \right) \sin 2\pi x/L - \left(\frac{L}{\pi}\right)^2 \frac{x}{4} \cos \frac{2\pi x}{L} \right]_0^L$$

$$= \frac{2}{L} \left\{ \frac{L^3}{6} - \left(\frac{L}{\pi} \frac{L^2}{4} - \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \right) \sin 2\pi + \left(\frac{L}{\pi}(0) - \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \right) \sin 0 \right.$$

$$\left. - \left(\frac{L}{\pi}\right)^2 \frac{L}{4} \cos 2\pi + \left(\frac{L}{\pi}\right)^2 (0) \cos 0 \right\}$$

$$\boxed{\langle x^2 \rangle = \left[\frac{1}{3} - \frac{1}{2\pi^2} \right] L^2} \quad = .2827\, L^2$$

(c) The momentum operator is $p_{op} = \frac{\hbar}{i} \frac{d}{dx}$: $\frac{d}{dx} \psi = \sqrt{\frac{2}{L}} \left(\frac{\pi}{L}\right) \cos \frac{\pi}{L} x$

$$\langle p \rangle = \int \psi^* \frac{\hbar}{i} \frac{d}{dx} \psi \, dx = \left(\frac{2}{L}\right) \frac{\hbar}{i} \frac{\pi}{L} \int_0^L \sin \frac{\pi}{L} x \cos \frac{\pi}{L} x \, dx$$

$$= \left(\frac{2}{L}\right) \frac{\hbar}{i} \left(\frac{\pi}{L}\right) \frac{1}{2} \left(\frac{L}{\pi}\right) \sin^2 \frac{\pi}{L} x \Big|_0^L = \frac{\hbar}{iL} \left[\sin^2 \pi - \sin^2 0 \right] = 0$$

$\boxed{\langle p \rangle = 0}$ as expected.

(d) $\langle p^2 \rangle = \int \psi^* \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi \, dx = -\hbar^2 \int \psi^* \frac{d^2}{dx^2}\psi \, dx$

$\frac{d^2}{dx^2}\psi = \sqrt{\frac{2}{L}} \frac{d^2}{dx^2} \sin\frac{\pi}{L}x = \sqrt{\frac{2}{L}} (-)\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi}{L}x$

$\langle p^2 \rangle = -\hbar^2 (-)\left(\frac{\pi}{L}\right)^2 \left[\frac{2}{L}\int_0^L \sin\frac{\pi}{L}x \, \sin\frac{\pi}{L}x \, dx\right]$

$= \hbar^2 \left(\frac{\pi}{L}\right)^2 \int \psi^*(x)\psi(x)\, dx = \hbar^2\left(\frac{\pi}{L}\right)^2 (1)$

$\boxed{\langle p^2 \rangle = \left(\frac{\hbar\pi}{L}\right)^2}$

(e) $\Delta p = \left[\langle p^2\rangle - \langle p\rangle^2\right]^{\frac{1}{2}} = \frac{\hbar\pi}{L}$

$\Delta x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{\frac{1}{2}} = \left[\frac{1}{3} - \frac{1}{2\pi^2} - \frac{1}{4}\right]^{\frac{1}{2}} L$

$\Rightarrow \Delta p \cdot \Delta x = \frac{\hbar\pi}{L}\left[\frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2}\right]^{\frac{1}{2}} L = \hbar\pi\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]$

$= \left[\frac{\pi^2}{12} - \frac{1}{2}\right]^{\frac{1}{2}}\hbar = \left[\frac{\pi^2 - 6}{12}\right]^{\frac{1}{2}}\hbar = \boxed{0.568\hbar}$