Homework 8 Solutions

7-2 (a) To the left of the step the particle is free with kinetic energy E and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x \le 0$$

To the right of the step the kinetic energy is reduced to $E-U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x \ge 0$$

with $D=0$ for waves incident on the step from the left. At $x=0$ both ψ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0) = A+B = C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A-B) = ik_2C.$$

(b) Eliminating C gives $A+B = \frac{k_1}{k_2}(A-B)$ or $A\left(\frac{k_1}{k_2}-1\right) = B\left(\frac{k_1}{k_2}+1\right)$. Thus,

$$R = \left|\frac{B}{A}\right|^2 = \frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

$$T = 1-R = \frac{4k_1k_2}{(k_1+k_2)^2}$$

(c) As $E \to U$, $k_2 \to 0$, and $R \to 1$, $T \to 0$ (no transmission), in agreement with the result for any energy $E<U$. For $E \to \infty$, $k_1 \to k_2$ and $R \to 0$, $T \to 1$ (perfect transmission) suggesting correctly that very energetic particles do not *see* the step and so are unaffected by it.

W-11) The reflection coefficient is $R = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$ where $k_1 = [2mE/\hbar^2]^{\frac{1}{2}}$ and $k_2 = [2m(E-U)/\hbar^2]^{1/2}$. We can rewrite this in terms of the ratio

$$r = k_2/k_1 = \left[\frac{E-U}{E}\right]^{\frac{1}{2}}$$

as

$$R = \left(\frac{1-r}{1+r}\right)^2 \qquad T = 1-R$$

(a) $E = 1.2U \Rightarrow$

$$r = \left[\frac{1.2U-U}{1.2U}\right]^{\frac{1}{2}} = 0.408$$

$$R = \left(\frac{1-.408}{1+.408}\right)^2 = 0.177 \quad \text{so} \quad \boxed{R = 0.177 \mid T = 0.823}$$

(b) $E = 2.0u \Rightarrow r = 0.707$ $\boxed{R = 0.029 \; ; \; T = 0.971}$

(c) $E = 10.0u \Rightarrow r = 0.949$ $\boxed{R = 6.8\times10^{-4} \; ; \; T = 0.9993}$

W-12) $\alpha = [2m(V_0 - E)/\hbar^2]^{\frac{1}{2}} = [2(mc^2)(V_0 - E)]^{\frac{1}{2}}/\hbar c$

(a) $E = 3eV$ $V_0 = 3.5eV$ $L = 0.5$ nm

$\Rightarrow$

$$\alpha L = \{[2(5.11\times10^5 eV)(0.5eV)]^{\frac{1}{2}}/197.3eV.nm\}\cdot(0.5nm) = 1.812$$

$$T = \left\{1 + \frac{1}{16}\left(\frac{(3.5eV)^2}{3eV\cdot 0.5eV}\right)\left[e^{1.812} - e^{-1.812}\right]^2\right\}^{-1} = \boxed{0.0523}$$

(b) $V_0 = 5eV \Rightarrow \alpha L_1 = 3.623$ $\boxed{T = 0.00273}$

Here the approximation $T \simeq [16E(V_0 - E)/V_0^2]\, e^{-2\alpha L}$ works pretty well.

(c) Changing to protons, $mc^2 = 938.3$ MeV

$$\alpha L_1 = \frac{[2(9.383\times10^8 eV)(2eV)]^{\frac{1}{2}}}{\hbar c}\cdot(0.5nm) = 155$$

$$T \simeq 16\frac{(5)(2)}{(3)(3)}e^{-2\alpha L_1} \sim e^{-310} \sim 10^{-135}$$ $\Rightarrow$ very small

W-13) The transmission probability for an α particle from a nucleus is given by Eq. (6.15). Using $r_0 = 7.25$ fm, $E_0 = 0.0993$ MeV and $R = 9$ fm we get the following results:

^{210}Po : $E = 5.40$ MeV , $Z = 84$ $\Rightarrow$

$$T = \exp\left[-4\pi(84)\sqrt{\frac{.0993}{5.40}} + 8\sqrt{\frac{(84)(9)}{7.25}}\right] = \exp[-143.14 + 81.69] = 2.06\times10^{-27}$$

Thus

$$t_{1/2} = \ln 2/fT = 0.693/(10^{20}/s)(2.06\times10^{-27}) \Rightarrow t_{1/2} = 3.37\times10^6\ \text{sec}$$

$\Rightarrow \boxed{t_{1/2} = 39 \text{ days}}$ The actual half-life is 138.4 days, so our estimate is too low by a factor of 3-4

^{214}Po : Here $E = 7.83$ MeV

$$\Rightarrow T = \exp\left[-4\pi(84)\sqrt{\frac{.0993}{7.83}} + 8\sqrt{\frac{(84)(9)}{7.25}}\right] = \exp\left[-118.87 + 81.69\right] = 7.12 \times 10^{-17}$$

$$t_{1/2} = 0.693 / (10^{20}/s)(7.12 \times 10^{-17}) = 9.73 \times 10^{-5}\,s \Rightarrow \boxed{t_{1/2} = 97.3\,\mu s}$$

The actual halflife is 164 μsec, so in this case our estimate is too low by about a factor of 2.

8-2 (a) $n_1 = 1, n_2 = 1, n_3 = 1$

$$E_0 = \frac{3\hbar^2\pi^2}{2mL^2} = \frac{3h^2}{8mL^2} = \frac{3(6.626 \times 10^{-34}\text{ Js})^2}{8(9.11 \times 10^{-31}\text{ kg})(2 \times 10^{-10}\text{ m})^2} = 4.52 \times 10^{-18}\text{ J} = 28.2\text{ eV}$$

(b) $n_1 = 2, n_2 = 1, n_3 = 1$ or
$n_1 = 1, n_2 = 2, n_3 = 1$ or
$n_1 = 1, n_2 = 1, n_3 = 2$

$$E_1 = \frac{6h^2}{8mL^2} = 2E_0 = 56.4\text{ eV}$$

8-3 $n^2 = 11$

(a) $$E = \left(\frac{\hbar^2\pi^2}{2mL^2}\right)n^2 = \frac{11}{2}\left(\frac{\hbar^2\pi^2}{mL^2}\right)$$

(b)

n_1	n_2	n_3	
1	1	3	
1	3	1	3-fold degenerate
3	1	1	

(c)
$$\psi_{113} = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{3\pi z}{L}\right)$$
$$\psi_{131} = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$
$$\psi_{311} = A\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

W-14) The wavefunctions must be solutions to the equation

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\psi = E\psi$$

and must go to zero at the edges of the well. The solutions all have the form

$$\psi = A \sin(n_1\pi x/L_x)\cdot \sin(n_2\pi y/L_y)$$

where n_1 and n_2 are any positive integers. The corresponding energy is

$$E = \frac{\hbar^2}{2m}\left[\frac{n_1^2\pi^2}{L_x^2}+\frac{n_2^2\pi^2}{L_y^2}\right]$$

Putting $L_x = L$ and $L_y = \frac{1}{2}L$ this reduces to $E = \left[n_1^2+4n_2^2\right]\frac{\pi^2\hbar^2}{2mL^2}$

Choosing various possibilities for n_1 and n_2 we get the following results

n_1	n_2	E
1	1	$5E_0$
2	1	$8E_0$
1	2	$17E_0$
2	2	$20E_0$
3	1	$13E_0$
3	2	$25E_0$
1	3	$37E_0$
2	3	$40E_0$
4	1	$20E_0$
4	2	$32E_0$

etc.

where $E_0 \equiv \frac{\pi^2\hbar^2}{2mL^2}$

so the energies of the 6 lowest states are

$$\boxed{5E_0,\ 8E_0,\ 13E_0,\ 17E_0,\ 20E_0,\ 20E_0}$$

8-10 $n=4$, $l=3$, and $m_l=3$.

(a) $L=[l(l+1)]^{1/2}\hbar=[3(3+1)]^{1/2}\hbar=2\sqrt{3}\hbar=3.65\times10^{-34}$ Js

(b) $L_z=m_l\hbar=3\hbar=3.16\times10^{-34}$ Js