Homework 9 Solutions

8-18 The state is $6g$

(a) $n = 6$

(b) $E_n = -\dfrac{13.6\text{ eV}}{n^2}$ $\qquad E_6 = \dfrac{-13.6}{6^2}\text{ eV} = -0.378\text{ eV}$

(c) For a g-state, $l = 4$

$$L = [l(l+1)]^{1/2}\hbar = (4\times 5)^{1/2}\hbar = \sqrt{20}\hbar = 4.47\hbar$$

(d) m_l can be −4, −3, −2, −1, 0, 1, 2, 3, or 4

$L_z = m_l\hbar$; $\cos\theta = \dfrac{L_z}{L} = \dfrac{m_l}{[l(l+1)]^{1/2}}\hbar = \dfrac{m_l}{\sqrt{20}}$

m_l	−4	−3	−2	−1	0	1	2	3	4
L_z	$-4\hbar$	$-3\hbar$	$-2\hbar$	$-\hbar$	0	$\hbar$	$2\hbar$	$3\hbar$	$4\hbar$
θ	153.4°	132.1°	116.6°	102.9°	90°	77.1°	63.4°	47.9°	26.6°

W-15) We can have $l = 0, 1, 2$, so the possibilities are

$(n, l, m) =$ (3,2,2) ; (3,2,1) ; (3,2,0) ; (3,2,−1) ; (3,2,−2)
(3,1,1) ; (3,1,0) ; (3,1,−1)
(3,0,0)

for a total of $\boxed{9\text{ states.}}$

8-22 $R_{2p}(r) = Are^{-r/2a_0}$ where $A = \dfrac{1}{2(6)^{1/2}a_0^{5/2}}$

$$P(r) = r^2R_{2p}^2(r) = A^2r^4e^{-r/a_0}$$

$$\langle r\rangle = \int_0^\infty rP(r)dr = A^2\int_0^\infty r^5e^{-r/a_0}dr = A^2a_0^6 5! = 5a_0 = 2.645\text{ Å}$$

W-16) $R(r) = Are^{-r/2a_0}$

So $\dfrac{d}{dr}R(r) = A\left[r\left(-\dfrac{1}{2a_0}\right)e^{-r/2a_0} + e^{-r/2a_0}\right] = A\left[1 - \dfrac{1}{2}\dfrac{r}{a_0}\right]e^{-r/2a_0}$

$$\frac{d^2}{dr^2}R(r) = A\left[-\frac{1}{2a_0}e^{-r/2a_0} + \left(1-\frac{1}{2}\frac{r}{a_0}\right)\left(-\frac{1}{2a_0}\right)e^{-r/2a_0}\right]$$

$$= A\left[\frac{r}{4a_0^2} - \frac{1}{a_0}\right]e^{-r/2a_0}$$

We need to solve

$$-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]R(r)+\frac{\ell(\ell+1)\hbar^2}{2mr^2}R(r)-\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{r}R(r)=E\,R(r)$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{r}{4a_0^2}+\frac{\hbar^2}{2m}\frac{1}{a_0}-\frac{\hbar^2}{2m}\left(\frac{2}{r}\right)\left(1-\frac{1}{2}\frac{r}{a_0}\right)\right]Ae^{-r/2a_0}+\frac{2\hbar^2}{2mr^2}Are^{-r/2a_0}$$

$$-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}Are^{-r/2a_0}=EAre^{-r/2a_0}$$

Cancell common factors of $A, r, e^{-r/2a_0}$ to get

$$-\frac{\hbar^2}{8ma_0^2}+\frac{\hbar^2}{2ma_0}\cdot\frac{1}{r}-\cancel{\frac{\hbar^2}{mr^2}}+\frac{\hbar^2}{2ma_0}\cdot\frac{1}{r}+\cancel{\frac{\hbar^2}{mr^2}}-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}=E$$

But $a_0=\left(\frac{4\pi\epsilon_0}{e^2}\right)\frac{\hbar^2}{m}$ so

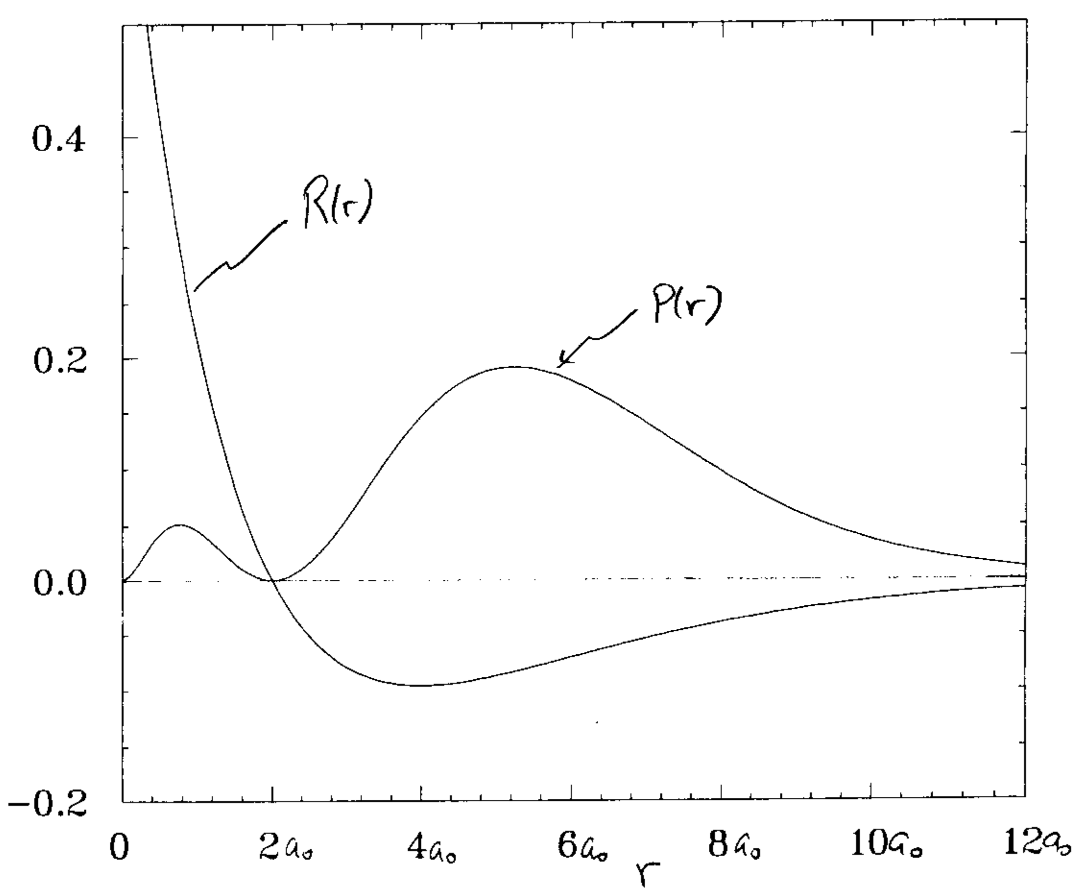
$$E=-\frac{\hbar^2}{8m}\left(\frac{e^2}{4\pi\epsilon_0}\frac{m}{\hbar^2}\right)^2+\cancel{\frac{\hbar^2}{m}\left(\frac{e^2}{4\pi\epsilon_0}\frac{m}{\hbar^2}\right)\frac{1}{r}}-\cancel{\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{r}}$$

$\Rightarrow$ the equation is solved if

$$\underline{E=-\frac{m}{8}\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar}\right)^2}$$

<u>W-17)</u> The wave function (see Table 8.3) is $R(r)=\left(\frac{1}{2a_0}\right)^{3/2}(2-r/a_0)\,e^{-r/2a_0}$

$\Rightarrow \quad P(r)=\left(\frac{1}{2a_0}\right)^3\left[4-4\,r/a_0+(r/a_0)^2\right]r^2e^{-r/a_0}$



W-18) The most probable value of r is found by looking for the peak of the probability distribution. We can find that by using $\frac{dP}{dr}=0$.
2p-State: For the 2p state $R(r) = A r e^{-r/2a_0}$ where A is a constant (see Table 8.3). Thus we have

$$P(r) = r^2 |R(r)|^2 = A^2 r^4 e^{-r/a_0}$$

$$\frac{dP}{dr} = A^2 \left\{ 4r^3 e^{-r/a_0} + r^4\left(-\frac{1}{a_0}\right)e^{-r/a_0} \right\} = A^2 r^3 \left[4 - \frac{r}{a_0}\right] e^{-r/a_0}$$

This is zero if

$$4 - \frac{r}{a_0} = 0 \Rightarrow \frac{r}{a_0} = 4 \Rightarrow \boxed{r = 4a_0}$$

2s-State: Here we have (see prob. W-17

$$P(r) = \left(\frac{1}{2a_0}\right)^3 \left[4 - 4\frac{r}{a_0} + \left(\frac{r}{a_0}\right)^2\right] r^2 e^{-r/a_0} = \left(\frac{1}{2a_0}\right)^3 \left[4r^2 - 4\frac{r^3}{a_0} + \frac{r^4}{a_0^2}\right] e^{-r/a_0}$$

$$\frac{dP}{dr} = \left(\frac{1}{2a_0}\right)^3 \left\{ \left[8r - 12\frac{r^2}{a_0} + 4\frac{r^3}{a_0^2}\right] e^{-r/a_0} + \left[4r^2 - 4\frac{r^3}{a_0} + \frac{r^4}{a_0^2}\right]\left(-\frac{1}{a_0}\right) e^{-r/a_0} \right\}$$

$$= \frac{1}{8}\frac{1}{a_0^2}\left[8\frac{r}{a_0} - 12\left(\frac{r}{a_0}\right)^2 + 4\left(\frac{r}{a_0}\right)^3 - 4\left(\frac{r}{a_0}\right)^2 + 4\left(\frac{r}{a_0}\right)^3 - \left(\frac{r}{a_0}\right)^4\right] e^{-r/a_0}$$

$$= \frac{1}{8}\frac{1}{a_0^2}\frac{r}{a_0}\left[8 - 16\frac{r}{a_0} + 8\left(\frac{r}{a_0}\right)^2 - \left(\frac{r}{a_0}\right)^3\right] e^{-r/a_0}$$

This is zero when

$$(8 - 16x + 8x^2 - x^3) = 0$$

where I used the abbreviation $x = \frac{r}{a_0}$. To find the roots of this equation we need to factor it. Notice from problem 4 that $\frac{dP}{dr}$ appears to be zero (P is a minimum) right at $r = 2a_0$. This suggests that one of the roots may be $x = 2$. Trying that, it is easy to see that the cubic polynomial can be factored as follows

$$(2-x)(4 - 6x + x^2) = 0$$

Thus the roots are $x = 2$ and $x = \frac{6 \pm \sqrt{36 - 4(4)}}{2} = 3 \pm \sqrt{5}$

From the graph in problem 4 it is clear that the actual overall maximum is at $x = 3 + \sqrt{5}$ $\Rightarrow$

$$\boxed{r = (3+\sqrt{5})a_0 = 5.236\, a_0}$$

W-19) (a) $V(r) = -\frac{e^2}{4\pi\epsilon_0} \cdot \frac{1}{r}$ So $\langle V \rangle = \langle -\frac{e^2}{4\pi\epsilon_0} \cdot \frac{1}{r} \rangle = -\frac{e^2}{4\pi\epsilon_0} \cdot \langle \frac{1}{r} \rangle$

$$\langle \tfrac{1}{r} \rangle = \int_0^\infty \frac{1}{r} P(r)\, dr = 4\left(\frac{1}{a_0}\right)^3 \int_0^\infty \frac{1}{r}\, r^2 e^{-2r/a_0}\, dr$$

$$= 4\left(\frac{1}{a_0}\right)^3 \int_0^\infty r e^{-2r/a_0}\, dr$$

As before, define $x = 2r/a_0 \Rightarrow r = \frac{1}{2}a_0 x \quad dr = \frac{1}{2}a_0\, dx \Rightarrow$

$$\langle \tfrac{1}{r} \rangle = 4\left(\frac{1}{a_0}\right)^3 \int_0^\infty \left(\tfrac{1}{2}a_0 x\right) e^{-x} \left(\tfrac{1}{2}a_0\right) dx = \frac{1}{a_0} \int_0^\infty x e^{-x} dx = \frac{1}{a_0}$$

Thus

$$\langle V \rangle = -\frac{e^2}{4\pi\epsilon_0} \cdot \frac{1}{a_0} = -(1.44\text{eV}\cdot\text{nm})/(0.0529\text{ nm})$$

$$\Rightarrow \boxed{\langle V \rangle = -27.2\text{ eV}}$$

(b) For the ground state the total energy is $E = -13.6\text{eV}$. Thus

$$\langle K \rangle + \langle V \rangle = E \Rightarrow \langle K \rangle = E - \langle V \rangle = -13.6\text{eV} - (-27.2\text{eV})$$

$$\Rightarrow \boxed{\langle K \rangle = +13.6\text{eV}}$$

It can be shown in general that for <u>any</u> state in hydrogen

$\langle V \rangle = 2E$ and $\langle K \rangle = -E = |E|$.