Inductance of Solenoid

(∞-long) piece of length l

n turns per length,

cross section area A

$$L = \frac{N\Phi_B}{I} = \boxed{\mu_0 n^2 l A}$$

proportional to volume $= lA$

and $\propto n^2$ - double turns:

turns doubles, flux thru each doubles $\rightarrow \times 4 \Rightarrow n^2$

Inductor w/ Current I

Changing $dI/dt \Rightarrow$

$$\mathcal{E} = L\, dI/dt$$

Power: $P = \mathcal{E} I$

$$= L\left(\frac{dI}{dt}\right) I =$$

$$P = LI\, dI/dt$$

Supply energy in dt:

$$dW = P\,dt$$

$$dW = LI\left(\frac{dI}{dt}\right)dt = LI\,dI$$

Work to increase Current $i = 0 \rightarrow I$

$$W = \int_0^I Li\,di = L\int_0^I i\,di = \frac{1}{2}LI^2$$

Inductor L w/ Current I

has stored energy $U_B = \frac{1}{2}LI^2$

analogous to energy stored

in a Capacitor C w/ charge q

$$U_E = \frac{1}{2}\frac{Q^2}{C}$$

Energy density of magnetic field = Energy / Volume

Energy stored in Solenoid

area A, length ℓ ⇒ Volume $= A\ell$

Uniform B ⇒ Energy is uniform

$\frac{\text{Energy}}{\text{Volume}} = \frac{U_B}{\text{Vol}} : u_B = \frac{U_B}{A\ell} = \frac{\frac{1}{2}LI^2}{A\ell}$

L for Solenoid $= \mu_0 n^2 A\ell$

$$u_B = \frac{\frac{1}{2}(\mu_0 n^2 A\ell) I^2}{A\ell} = \frac{1}{2}\mu_0 n^2 I^2$$

Inside Solenoid

$B = \mu_0 I n$ ⇒

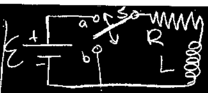
$I = B / \mu_0 n$

$$u_B = \frac{1}{2}\mu_0 n^2 \left(\frac{B}{\mu_0 n}\right)^2$$

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0}$$ energy per unit Volume

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

Circuit w/ R and L



RL Circuit

Close switch → current I in Resistor increases

w/o L ⇒ $I = \mathcal{E}/R$

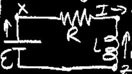
w/ L: ⇒ self-induced $\mathcal{E}_L$

Lenz ⇒ opposes increase in I

⇒ opposes battery $\mathcal{E}$

Resistor sees 2 $\mathcal{E}$'s: $\mathcal{E}$ (battery), $\mathcal{E}_L$ Inductor

$\mathcal{E}_L = -L\frac{dI}{dt}$. Switch at position a:



Loop Theorem x→y→z→x

x→y: $V_R = -IR$

y→z: $V_L = -L\,dI/dt$

z→x: $\mathcal{E}$

$$-IR - L\frac{dI}{dt} + \mathcal{E} = 0$$

$$\mathcal{E} = L\frac{dI}{dt} + IR$$

$$\frac{dI}{dt} = \frac{\mathcal{E}-IR}{L} = \frac{\mathcal{E}}{L} - \frac{R}{L}I$$

When close switch ($t=0$)

$$I=0 \Rightarrow \left.\frac{dI}{dt}\right|_{t=0} = \frac{\mathcal{E}}{L}$$

at $t=\infty \Rightarrow dI/dt = 0$

$$0 = \frac{\mathcal{E}}{L} - \frac{R}{L}I \Rightarrow I = \mathcal{E}/R$$

$$I = \frac{\mathcal{E}}{R}\left(1-e^{-Rt/L}\right)$$

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L} - \frac{R}{L}I = \frac{\mathcal{E}}{L} - \frac{\mathcal{E}}{L} + \frac{\mathcal{E}}{L}e^{-Rt/L}$$

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L}e^{-Rt/L}$$

is derivative of I when plugged in

Capacitance: $e^{-t/RC}$

⇒ Characteristic time: $t_c = RC$

Inductance: $e^{-Rt/L}$

⇒ Characteristic time: $t_L = L/R$ dimensions of time

Plug in $t_L = L/R$

$I = \frac{\mathcal{E}}{R}(1-e^{-1}) = .63\frac{\mathcal{E}}{R}$

time at which current rises to 63% of final value.

After long time.. open switch to position ⓑ $\Rightarrow \mathcal{E} = 0$

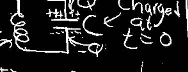
Loop: $L\frac{dI}{dt} + IR = 0$

$I = \frac{\mathcal{E}}{R}e^{-t/t_L} = \frac{\mathcal{E}}{R}e^{-Rt/L}$

t_L = time for current to decay to 37% of it's original value.

energy to maintain current when $\mathcal{E} = 0$ is energy stored in magnetic field

L-C Circuit



$I = 0$ at $t = 0$

at $t=0$ Energy stored in C

$U_E = \frac{1}{2}\frac{Q^2}{C}$, $U_B = \frac{1}{2}LI_s^2 = 0$

Capacitor discharges thru Inductor $L \Rightarrow$ current I from $\frac{dQ}{dt} = I$, as charge in C decreases, U_E decreases, current in L increases $\Rightarrow$ U_B increases $\Rightarrow Q = 0$ an

$U_E = \frac{1}{2}\frac{Q^2}{C} = 0$, $U_B = \frac{1}{2}LI^2$

Oscillating system with frequency f

$\omega = 2\pi f$ analgous to mass on spring

mass m, spring const. k

Charge q_h corresponds to displacement X

Current $I = \frac{dq_h}{dt}$ Corres: $V = \frac{dX}{dt}$

C corres. to $\frac{1}{k}$, L corres to m

Spring: $U = \frac{1}{2} k X^2$

$U_E = \frac{1}{2}\left(\frac{1}{C}\right) q_h^2$ ✓

mass: $KE = \frac{1}{2} m V^2$

$U_B = \frac{1}{2} L I^2$ ✓

frequency $\omega = \sqrt{\frac{k}{m}}$

LC circuit: $\omega = \frac{1}{\sqrt{LC}}$

Total Energy $U = U_B + U_B$

$U = \frac{1}{2} L i^2 + \frac{1}{2} \frac{q_h^2}{C}$. Solve for i.

at $t = 0$ $q_h = Q$, $i = 0$

$U = \frac{1}{2} L i^2 + \frac{1}{2} \frac{q_h^2}{C} = \frac{Q^2}{2C}$

Energy Cons. (no R)

True for all time:

$i = \sqrt{\frac{1}{LC}} \sqrt{Q^2 - q_h^2}$ Reminds:

$V = \pm \omega \sqrt{A^2 X^2}$

$V = \frac{dx}{dt} \Leftrightarrow i = \frac{dq_h}{dt}$

$x = A\cos\omega t \Rightarrow$

$q_h = Q\cos\omega t$

Where $\omega = \frac{1}{\sqrt{LC}}$

$q_h = Q\cos\sqrt{\frac{1}{\sqrt{LC}}}\,t$

If we add resistance

$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q_h^2}{C}$

$\frac{dU}{dt} = -i^2R \leftarrow$ Power Loss in R

$Li\frac{di}{dt} + \frac{q_h}{C}\frac{dq_h}{dt} = -i^2R$

$i = \frac{dq_h}{dt} \Rightarrow \frac{di}{dt} = \frac{d^2q_h}{dt}$

Divide by $i \Rightarrow$

$L\frac{d^2q_h}{dt^2} + R\frac{dq_h}{dt} + \frac{q_h}{C} = 0$

$q_h = Q e^{-Rt/2L}(\cos\omega' t + \phi)$

$\omega' = \sqrt{\omega^2 - (R/2L)^2}$

$\omega = \frac{1}{\sqrt{LC}}$