Sample questions P711 Fall 2014

1. A particle of mass $m$ moves in the spherically symmetric potential $V(r)$. Treat motion in a plane with plane-polar coordinates $r, \vartheta$.

   (a) What are the conserved quantities?
   (b) Treat motion that is close enough to circular motion $r = r_0$ that the deviations $\eta(t) = r(t) - r_0$ can be evaluated in “first order.” Solve for the motion $\eta(t), \vartheta(t)$.
   (c) The initial conditions for the motion in (b) are given as $r_i, \dot{r}_i, \vartheta_i, \dot{\vartheta}_i$. Find the relation between these parameters and the terms in the solution obtained in (b).

2. Treat the longitudinal (1D motion along $\hat{x}$) of the system of three masses $M, m, M$ linked by springs $k$, $M > m$.

   (a) Determine the normal mode frequencies of the system.
   (b) If the middle mass $m$ is driven by a force $F_0 \cos \omega_0 t \hat{x}$ determine its displacement $x_m(t)$, including the effects of coupling to the masses $M$. For $\omega_0 = 2\sqrt{k/m}$, determine the phase of its motion relative to the driving force.

3. For a heavy symmetrical top with one point fixed, the Lagrangian can be written in terms of 3 Euler angles $\vartheta, \varphi$ and $\psi$:

   \[ L = \frac{I_1}{2}(\dot{\vartheta}^2 + \varphi^2 \sin^2 \vartheta) + \frac{I_2}{2}(\dot{\varphi} + \varphi \cos \vartheta)^2 - Mg\ell \cos \vartheta \]

   The constants $I_1$ and $I_2$ are “principal moments of inertia,” $g$ is the gravitational acceleration; the center of mass is at distance $\ell$ from the tip.

   Construct and interpret three (3) constants of the motion.

4. A plane circular loop of rope of total mass $M$ and radius $R$ is rotating about its axis in free space. As the rope spins faster, it stretches but remains circular and in the plane. The radius increases over its unstretched radius $R_0$ and the stretching energy is

   \[ V = \frac{1}{2}k(2\pi)^2(R - R_0)^2 \]
(a) Write a Lagrangian that governs this motion.
(b) Identify two constants of the motion.
(c) Write the equation of motion for $R(t)$. (You need not solve it.)
(d) Find the equilibrium radius as a function of the angular momentum.

5. Three identical point masses $M$ are arranged in a straight line along $\hat{x}$ and nearest neighbor pairs are joined by springs $K$. Find the normal mode frequencies for the $\hat{x}$-motion and sketch the normal modes.

6. The Planck satellite is located at the Lagrange L2 point. In leading approximation, the arrangement is that the sun $S$, earth $E$ and satellite $P$ are aligned in a straight line $S-E-P$ and all perform circular motion with angular frequency $\omega$ about the center-of-mass of the earth-sun pair.

(a) Set up the force balance equation for the Planck satellite to maintain its position relative to the earth and sun, i.e., that its orbital frequency be the same as that of the earth.
(b) With $M_S = 1.987 \times 10^{30}$ kg, $M_E = 5.97 \times 10^{24}$ kg, and $R_{ES} = 1.495 \times 10^{11}$ m, determine the distance $r$ of the satellite from the earth.

7. In an elastic collision of two equal rest-masses $M$, initially mass 2 is at rest at the origin $r = 0$ and mass 1 is incident from the left, along the $x$-axis with momentum $p = p\hat{x}$. Treat the final case in which the outgoing masses both move at an angle $\vartheta/2$ relative to the $\hat{x}$-axis.

(a) If the motion is nonrelativistic, give an explicit value for the angle $\vartheta$ between the outgoing masses.
(b) If the motion is relativistic, give an expression for the angle between the outgoing masses and evaluate it in the limit $p \gg Mc$.

8. (Inverse Compton scattering) A photon of frequency $\nu$, wavelength $\lambda = c/\nu$, energy $h\nu$, momentum $h\nu/c$ has a head-on collision with an electron of total energy $E$ and momentum $p$. The photon is backscattered (scattering angle $\pi$) Find the energy of the final photon in terms of the given quantities.
9. An earth satellite is launched into what is intended to be a circular orbit of altitude 120 km (at distance 6500 km from [spherical] earth center). At burnout the satellite is at the correct altitude and in level flight but its velocity exceeds circular velocity by one percent. What is the maximum altitude of the satellite in the resulting elliptical orbit?

10. The $\pi^0$ meson has rest mass energy $mc^2 = 135.1$ Mev and has no charge and no spin and moves along $\hat{x}$ with speed $\beta c$. It decays into two $\gamma$ rays (quanta with no rest mass and momentum $p = \epsilon/c$). If one photon is observed at angle $\theta$ relative to the initial motion, determine the angle of the other photon and the energies of both photons.

11. By how much must the energy of an incident photon exceed the deuteron binding energy in order to photo-disintegrate an initially stationary deuteron? 

$$\text{photon} + d \rightarrow n + p$$

The answer is to include relativistic effects. [proton rest mass energy 938.27 MeV; neutron 939.56 MeV; deuteron 1875.61 MeV, binding 2.22 MeV]

12. Two particles of masses $M_1$ and $M_2$ at $r_1$ and $r_2$ interact by the central pair potential, $r = |r_1 - r_2|$. 

$$u(r) = \epsilon[(R/r)^{12} - 2(R/r)^6]$$

(a) Construct the Lagrangian for the relative motion, in the variable $r = r_1 - r_2$.

(b) What quantities are conserved in the relative motion?

(c) For what ranges of energy is the relative motion bounded? Explain.

13. A uniform string with fixed ends has total length $\ell$, mass density $\sigma$ and uniform tension $\tau$. A point mass $m$ is attached at the center. The string is stretched along the $\hat{x}$-axis from $-\ell/2$ to $\ell/2$ and has small amplitude vibrations along the $\hat{y}$-axis.

(a) Find the equations of motion for the two halves of the string and for the point mass.
(b) Show that there are normal modes in which the point mass does not move.

(c) Derive an expression for the frequencies of normal modes in which the point mass does move.

14. State Hamilton’s principle. Evaluate the action integral for a free particle of mass $M$ moving between positions $r_1$ and $r_2$ in time $t$.

15. Describe the distinction between mechanical momentum and canonical momentum for a non-relativistic charged particle (mass $m$ and charge $q$) in an electromagnetic field with scalar and vector potentials $\phi$ and $A$. Is the transformation linking the mechanical and canonical momenta in this example a canonical transformation? Explain.