Lecture 13: Oscillations I (1 Oct 14)

O. review 29 Sep homework

A. Small amplitude expansion

1. FW Sec. 21 starts with 1D picture (expand about a position of zero force) and then to quadratic forms for expansion of both kinetic energy and potential energy:

\[ T = \sum_j \frac{m_j}{2} \dot{x}_j^2 = \sum_j \frac{m_j}{2} \left( \sum_i \frac{\partial x_j}{\partial q_i} \dot{q}_i \right)^2 = \frac{1}{2} \sum_{ij} T_{ij} \dot{q}_i \dot{q}_j \]

where \( T_{ij} = T_{ji} \) (if not seen immediately, juggle the dummy indices) \( T \) must be a positive definite matrix, from its role in the (positive) kinetic energy. Then with the \( q_i \) being increments relative to a position with zero gradients:

\[ \Delta \Phi = \frac{1}{2} \sum_{ij} \Phi_{ij} q_i q_j \]

The matrix \( \Phi_{ij} \) must be positive definite, \( x^T \cdot \Phi \cdot x > 0 \) for stable oscillations but that is not automatic for expansion about an arbitrary point.

2. The Lagrange equations then become:

\[ \sum_j T_{ij} \ddot{q}_j = -\sum_j \Phi_{ij} q_j \]

3. These are coupled equations. Seek oscillatory solutions: (1) For several displacements handle the specification of magnitude and phase by complex amplitude representation: \( \delta q_j = A_j \exp(-i \omega t + \phi_j) \). (2) The coupled equations are homogeneous linear equations, with special (characteristic, eigen- ) frequencies.
B. Eigenvalue problems

1. The math problem at this stage is closely related to the eigenvalue problem of finding solutions of

\[ \mathbf{M} \cdot \mathbf{v} = \lambda \mathbf{v} \]

for real symmetric matrices \( \mathbf{M} \). To begin, allow \( \mathbf{v} \) to be complex vector.

2. First prove the allowed \( \lambda \) are real: (\( \mathbf{v}^\dagger \) denotes complex-conjugate-transpose)

\[ \mathbf{v}^\dagger \cdot \mathbf{M} \cdot \mathbf{v} = (\mathbf{M} \cdot \mathbf{v})^\dagger \cdot \mathbf{v} = [\mathbf{v}^\dagger \cdot \mathbf{M} \cdot \mathbf{v}]^* \]

and then see that the \( \mathbf{v} \) can be chosen real. (Nondegenerate and degenerate cases)

3. Orthogonality of eigenvectors follows.

4. The eigenvectors form a complete orthonormal set (a basis) and can be used for expansions in the “space.”

5. The Lagrange equations from (A) were:

\[ \omega^2 \sum_j T_{ij} q_j^0 = \sum_j \Phi_{ij} q_j^0 \]

with real symmetric matrices \( T_{ij} \) and \( \Phi_{ij} \).

6. This has the form of a generalized eigenvalue problem \( \lambda = \omega^2 \)

\[ \lambda \mathbf{T} \cdot \mathbf{q} = \mathbf{M} \cdot \mathbf{q} \]

Theorems that \( \lambda \) is real carry over to this form, but the orthogonality relation is

\[ \mathbf{q}_i^\dagger \cdot \mathbf{T} \cdot \mathbf{q}_i = t_i \delta_{ij} \]

7. The text at pp.91-95 develops this more general formulation. However, the most useful algorithms for numerical work assume \( T_{ij} \) is diagonal. The process of bringing this matrix to diagonal form amounts to transforming from a non-orthogonal basis and has serious numerical instabilities. It is worth the effort, generally, to set up the problem in a representation where \( \mathbf{T} \) is diagonal.
8. In the “clean case” of diagonal $T$, one can scale out unequal masses and have the unit matrix. Then the eigenvectors of $M \cdot v_j = \lambda_j v_j$ are simply orthogonal and can be used to bring $M$ to diagonal form by

$$u_i = \sum_k C_{ik} v_k; \quad u_i^\dagger M_{ij} u_j = \sum_{ij} v_i^\dagger v_j \sum_k C_{ik} C_{jk} M_{ij}$$

9. Text develops the modal matrix formed from eigenvectors when $T$ is not diagonal and then uses it to transform from the original coordinates $u_j$ to the “normal coordinates” (here $v_k$) that diagonalize the Lagrangian.

C. Example – coupled pendulums

1. FW Sec 23. Two identical pendulums of length $\ell$ and hanging masses $m$ coupled by a spring $\Phi = (K/2)(x_1 - x_2)^2$. [In a plane]

2. Use the angles $\theta_i$; the lateral displacements are $\delta x_i = \ell \sin \theta_i \simeq \ell \theta_i$. The gravitational potential energy measured relative to “straight down” is

$$mg\ell (1 - \cos \theta_i) \simeq mg\ell \theta_i^2 / 2$$

3. The kinetic energy is “diagonal”

$$T = \frac{m}{2} \ell^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

and the total potential energy is (“small angles”)

$$\Phi = \frac{mg\ell}{2} (\theta_1^2 + \theta_2^2) + \frac{K\ell^2}{2} (\theta_1 - \theta_2)^2$$

4. Lagrange equations with $L = T - \Phi$ are

$$m\ell^2 \ddot{\theta}_1 + (mg\ell)\theta_1 + K\ell^2 (\theta_1 - \theta_2) = 0$$

$$m\ell^2 \ddot{\theta}_2 + (mg\ell)\theta_2 + K\ell^2 (\theta_2 - \theta_1) = 0$$

Ad hoc solution: add and subtract the equations:

$$m\ell^2 \frac{d^2}{dt^2} (\theta_1 + \theta_2) + (mg\ell)(\theta_1 + \theta_2) = 0$$
\[
m \ell^2 \frac{d^2}{dt^2} (\theta_1 - \theta_2) + (mg\ell + 2K\ell^2)(\theta_1 - \theta_2) = 0
\]
to get frequencies (lower frequency when swing together):
\[
\omega_+^2 = \frac{g}{\ell}; \omega_-^2 = (\frac{g}{\ell}) + (2K/m)
\]

5. More systematic, define \( \alpha = K/m \) \( \beta = g/\ell \) and look for the normal modes \( \theta_j \propto \exp(-i\omega t) \). The equations to be solved are
\[
\omega^2 \theta_1 = \beta \theta_1 + \alpha (\theta_1 - \theta_2)
\]
\[
\omega^2 \theta_2 = \beta \theta_2 + \alpha (\theta_2 - \theta_1)
\]

6. These become the pair of homogeneous equations
\[
[\beta + \alpha - \omega^2] \theta_1 - \alpha \theta_2 = 0
\]
\[
-\alpha \theta_1 + [\beta + \alpha - \omega^2] \theta_2 = 0
\]

7. Determinental equation for the coefficients give
\[
[\beta + \alpha - \omega^2]^2 - \alpha^2 = 0 \Rightarrow [\beta + \alpha - \omega^2] = \pm \alpha
\]

8. These are the two normal modes that were seen by inspection: (1) \( \omega^2 = \beta \), with \( \theta_1 = \theta_2 \) and (2) \( \omega^2 = \beta + 2\alpha \) with \( \theta_1 = -\theta_2 \).

9. Set up the initial value problem in terms of the normal mode coordinates \( \xi_1 = (1, 1)/\sqrt{2} \) and \( \xi_2 = (1, -1)/\sqrt{2} \).
\[
\theta_1 = (1/\sqrt{2})[\xi_1(t) + \xi_2(t)]; \theta_2 = (1/\sqrt{2})[\xi_1(t) - \xi_2(t)]
\]
with \( \xi_j(t) = A_j \cos(\omega_j t + \delta_j) \). If initially static with displacements \( \vartheta_i \),
\[
\vartheta_1 = (1/\sqrt{2})[A_1 + A_2], \text{etc.}
\]