Lecture 17: Continuum Limit II (10 Oct 14)

Mid-term exam, take-home, due Monday 13 Oct. at 5 PM; no class Monday.

A. Continuum approximation for discrete chain

1. Chain of longitudinal oscillators

\[ L = T - V = \frac{m}{2} \sum_j \dot{\eta}_j^2 - \frac{K}{2} \sum_j (\eta_{j+1} - \eta_j)^2 \]

\[ m\ddot{\eta}_j = -K[2\eta_j - \eta_{j+1} - \eta_{j-1}] \]

string of tension \( \tau \), unit length \( a \); small amplitude transverse vibration.

\[ m\ddot{y}_j = \tau[\sin \theta_{j,j+1} - \sin \theta_{j-1,j}] \approx (\tau/a)[y_{j+1} + y_{j-1} - 2y_j] \]

2. Construct solutions: \( y_j, \eta_j = y_0 \exp(i[qa - \omega(q)t]) \)

\[ \omega(q)^2 = 2(K/m)[1 - \cos(qa)], \quad K/m \leftrightarrow \tau/(ma) \]

3. Masses in the chain: successive displacements have relative phase \( \exp(iqa) \);

\( x \) coordinate \( x_j = ja \). Wavenumber to wavelength by \( q = 2\pi/\lambda \).

Smooth variation for \( qa << 1 \Leftrightarrow \lambda >> a \).

4. Chain of spacing \( a \), \( N \) atoms in repeat unit (periodic boundary conditions).

“Shortest wavelength” is \( \lambda = a \), longest is \( \lambda = Na \). \( \lambda \) for the periodic boundary condition \( qa = n2\pi/N \) with \( n \) cycles in length \( Na \)

is \( \lambda = Na/n = 2\pi/q \), i.e., \( q \) is the conventional wave number.

5. Continuum limit of the discrete chain equations: wavelength much longer than the spacing \( \lambda >> a \):

\[ qa << 1 \Rightarrow \omega(q) \approx \sqrt{K/m} \ q a \]

6. Continuum form of equations of motion (I): approximate the dynamics of the discrete chain. Central difference to get finite difference approximation to the second derivative:

\[ f(x + h) + f(x - h) - 2f(x) \approx \frac{h^2 d^2 f}{dx^2} \]
and use \( x = ja \) and “smooth variation” of displacement \( \eta \) at \( x \)

\[
\eta_{j+1} + \eta_{j-1} - 2\eta_j \simeq d^2\eta/dj^2 \rightarrow a^2 \partial^2\eta/\partial x^2
\]

Define the mass density \( \rho = m/a \) and a tension \( \tau = Ka \)

\[
\rho \partial^2\eta/\partial t^2 = \tau \partial^2\eta/\partial x^2
\]

7. Continuum form (II): directly approximate the equation of motion for transverse vibration of string element \( dx \) with mass density \( \rho \), tension \( \tau \) and displacement \( y(x,t) \).

\[
\rho dx \partial^2 y/\partial t^2 = \tau [\partial y/\partial x|_{x+(dx/2)} - \partial y/\partial x|_{x-(dx/2)}] \simeq \tau dx \partial^2 y/\partial x^2
\]

8. Either way, the equations of motion become, with density \( \rho = m/a \) and tension \( \tau = Ka \)

\[
\rho \partial^2\xi/\partial t^2 = \tau \partial^2\xi/\partial x^2
\]

\[
Ka^2/m = \tau/\rho \equiv v^2
\]

9. The equation of motion is a 1D wave equation, general solutions

\[
\xi(x,t) = f(x-vt) + g(x+vt)
\]

and special class: harmonic waves

\[
\xi(x,t) = A \sin([kx \pm \omega t] + \delta); \omega/k = v = \sqrt{\tau/\rho}
\]

B. Fourier series solutions

1. Harmonic waves subject to boundary conditions at \( x = 0, x = \ell \).
\( \xi \propto Y(x) \exp(-i\omega t) \) to enable satisfying boundary conditions at all times \( t \).

\[
-\omega^2 Y = v^2 d^2Y/dx^2 \rightarrow Y = \sin(kx + \delta); \omega^2 = v^2 k^2
\]
2. This is again an eigenvalue problem, now as a differential equation. The boundary conditions specify the \( k \) values and hence the angular frequencies \( \omega \).

3. Fixed ends: \( Y(0) = Y(\ell) = 0 \) \( \Rightarrow \delta = 0; \) \( k\ell = n\pi, Y_n(x) = \sin(n\pi x/\ell), n = 1, 2, 3, \ldots \). Now, though, there is no upper limit to \( n \), no largest \( k \), no shortest \( \lambda \).

4. The solutions \( Y_n \) are orthogonal and can be normalized; a complete orthonormal set (i.e., a basis):

\[
\int_0^\ell dx \sin(n_1 \pi x/\ell) \sin(n_2 \pi x/\ell) = \frac{1}{2} \delta_{n_1,n_2} \]

5. Expand the general solution satisfying these boundary conditions:

\( Y(x,t) = \sum_n A_n \sin(k_n x) \cos(\omega_n t + \delta_n) \);

\( Y(x,0) = \sum_n A_n \sin(k_n x) \cos \delta_n; \ \dot{Y}(x,0) = \sum_n A_n \sin(k_n x)(-\omega_n) \sin \delta_n \)

and the coefficients in the series are obtained using the orthogonality:

\[
f(x) = \sum_n C_n \sin(k_n x) \rightarrow C_n = \frac{(2/\ell)}{\int_0^\ell dx f(x) \sin(k_n x)} \]

C. Continuum Lagrangian; Hamilton’s principle

1. FW Sec. 25

2. The Lagrangian for the discrete chain (spacing \( a \)) is re-written:

\[
L = \frac{m}{2a} \sum_j a \dot{\eta}_j^2 - \frac{Ka}{2} \sum_j a \frac{1}{a^2} [\eta_j - \eta_{j-1}]^2
\]

Replace the first-difference by a spatial derivative; replace the sum on \( j \) by an integral over the length of the string \( \delta x = a \delta j (\delta j = 1); \delta x \rightarrow dx: \)

\[
L = \int_0^\ell L dx; L = \frac{\rho}{2} (\frac{\partial \eta}{\partial t})^2 - \frac{\tau}{2} (\frac{\partial \eta}{\partial x})^2
\]
\[ \sum \rightarrow f \] for smooth summands – a frequent approximation for “large” systems, error estimates available.

3. FW p. 127 With the eigenfunctions \( \rho_n(x) \) satisfying \( d^2 \rho_n(x)/dx^2 = -k_n^2 \rho_n(x) \) (and having a square root of density in the normalization) write the general wave motion as a sum of normal coordinate terms \( (\omega_n = v k_n) \)

\[
\begin{align*}
u(x,t) &= \sum_{n=1}^{\infty} \rho_n(x) \zeta_n(t) \\
\Rightarrow L &= \frac{1}{2} \sum_{n=1}^{\infty} (\dot{\zeta}_n^2 - \omega_n^2 \zeta_n^2)
\end{align*}
\]

4. The equation of motion then follows as a special case of generalizing Hamilton’s principle to cover the Lagrangian density \( L \)

\[
L = \int dx \mathcal{L}(u, \partial u/\partial x, \partial u/\partial t)
\]

\[
\delta \int_{t_1}^{t_2} dt \int_0^\ell dx \mathcal{L}(u, \partial u/\partial x, \partial u/\partial t) = 0
\]

5. Let \( u \rightarrow u + \delta u \), subject to \( \delta u(0,t) = \delta u(\ell,t) = 0 \) [all \( t \), fixed ends] and \( \delta u(x,t_1) = \delta u(x,t_2) = 0 \) [all \( x \)]. The stationary condition (the factor with \( \delta u(x,t) \)) is

\[
\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial u/\partial t)} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial u/\partial x)} - \frac{\partial \mathcal{L}}{\partial u} = 0
\]

6. Confirm that the equation of motion for the continuum chain (string) is recovered, \( u \leftrightarrow \eta \).

7. Later: will construct the Hamiltonian density – with a definition for the generalized momentum.