Lecture 21: Rigid Body Rotations (20 Oct 14)

A. Rotation–moment of inertia

1. FW Sec. 26; L & L Secs. 32, 33.

2. build on material of FW Secs. 6, 7, L6, 7.

3. The relation between inertial and body frame time derivatives of a vector $s$ is

$$\frac{ds}{dt}_{\text{inertial}} = \frac{ds}{dt}_{\text{body}} + \tilde{\omega} \times s$$

4. In particular for the position $r$, the relation between inertial and body frame time derivatives becomes

$$v = \frac{dr}{dt}_{\text{inertial}} = \frac{dr}{dt}_{\text{body}} + \tilde{\omega} \times r$$

5. Start with case of no net translation, but with a rotation relative to an inertial frame.

6. Then if the vector $r$ is fixed in the body frame coordinates, the total (rotational) kinetic energy is

$$T = \frac{1}{2} \sum_p m_p v_p^2 = \frac{1}{2} \sum_p m_p (\tilde{\omega} \times r_p) \cdot (\tilde{\omega} \times r_p)$$

7. Use vector cross product identities $[a \cdot (b \times c) = (a \times b) \cdot c]$:

$$(\tilde{\omega} \times r) \cdot (\tilde{\omega} \times r) = [(\tilde{\omega} \times r) \times \tilde{\omega}] \cdot r = -[\tilde{\omega} \times (\tilde{\omega} \times r)] \cdot r = \omega^2 r^2 - (\tilde{\omega} \cdot r)^2 = \omega^2 r_{\perp}^2$$

(Check: $| (\tilde{\omega} \times r) | = \omega r \sin \theta$, $\tilde{\omega} \cdot r = \omega r \cos \theta$)

8. Define the moment of inertia tensor (Cartesian components)

$$I_{ij} \equiv \sum_p m_p (r_p^2 \delta_{i,j} - x_{pi} x_{pj})$$
9. The rotational kinetic energy is
\[ T = \frac{1}{2} \vec{\omega} \cdot I \cdot \vec{\omega} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j \]

10. The angular momentum is \( \vec{L} = I \cdot \vec{\omega} \), using
\[ \vec{L}_i = \sum_p [\vec{r}_p \times (m_p \vec{\omega} \times \vec{r}_p)]_i = \sum_p m_p [\vec{\omega} \cdot \vec{r}_p^2 - (\vec{\omega} \cdot \vec{r}_p) \vec{r}_p] \]
and the rotational kinetic energy is \( T = \frac{1}{2} \vec{\omega} \cdot \vec{L} \).

11. Parallel axis theorem – relative to the center of mass. Say the positions \( \vec{r}_p \) are measured relative to the center of mass, so
\[ \sum_p m_p \vec{r}_p = 0 \]
Then calculate the moment of inertia tensor \( \tilde{I} \) relative to a new vector position \( \vec{a}, \vec{s}_p = \vec{r}_p - \vec{a} \). The \( r, a \) cross-terms in the expanded inertia tensor all vanish because, e.g.,
\[ \sum_p m_p r_{pi} a_j = a_j \sum_p m_p r_{pi} = 0 \]
and so there is the simplification
\[ \tilde{I}_{ij} = I_{ij} + (\sum_p m_p)(a^2 \delta_{i,j} - a_i a_j) \]

12. The moment of inertia tensor \( I_{ij} \) is a real symmetric matrix and has an eigenvalue/eigenvector problem
\[ I \cdot \vec{x}_j = I_j \vec{x}_j \]
The eigenvectors are orthogonal (“principal axes”) and the \( I_j \) are real and positive (for stability). Symmetric mass distributions reduce the number of distinct \( I_j \) (2 for axial symmetry and 1 for spherical symmetry).

13. The eigenvalues satisfy \( I_a + I_b \geq I_c \) for any choice of 3 distinct \( a, b, c \).

[LL Sec.32] Proof uses the fact that
\[ I_a = \vec{x}_a \cdot I \cdot \vec{x}_a = \sum_p m_p r_{p,a}^2 \]
B. General motion

1. The configuration of the rigid body is specified by 6 coordinates: 3 for the center of mass and 3 angles (spherical polars for the body axis plus a rotation about that axis) that give the orientation of the internal axes of the rigid body.

2. Let $\mathbf{R}$ be the center of mass position and $\mathbf{s}$ be the position taken relative to the center of mass, $\mathbf{r} = \mathbf{R} + \mathbf{s}$, with (bold face denotes vectors here)

$$\sum_p m_p \mathbf{s}_p = 0$$

3. The coordinates in the center-of-mass frame may be taken relative to a body-fixed set of axes. So

$$\frac{d\mathbf{r}}{dt}_{\text{inertial}} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}}{dt}_{\text{inertial}} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}}{dt}_{\text{body}} + \mathbf{\omega} \times \mathbf{s}$$

4. Let $\dot{s}_p$ be the velocity in cm frame relative to an inertial coordinate frame. Use:

$$\sum_p m_p (\dot{\mathbf{R}} + \dot{s}_p)^2 = \dot{\mathbf{R}}^2 \sum_p m_p + 2\dot{\mathbf{R}} \cdot \sum_p m_p \dot{s}_p + \sum_p m_p \dot{s}_p^2$$

On the right-hand-side, the first sum gives the total mass $M$, the second sum adds to zero.

5. Thus the kinetic energy is a sum of cm translational motion and “internal” kinetic energy $T'$:

$$T = \frac{M}{2} \dot{\mathbf{R}}^2 + T'$$

6. The angular momentum sum is [2 cross-term sums vanish]

$$\mathbf{L} = \sum_p m_p (\mathbf{R} + \mathbf{s}_p) \times \left( \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}_p}{dt}_{\text{inertial}} \right) = M\mathbf{R} \times \frac{d\mathbf{R}}{dt} + \sum_p m_p \mathbf{s}_p \times \frac{d\mathbf{s}_p}{dt}_{\text{inertial}}$$

$$\equiv \mathbf{L}_{\text{cm}} + \mathbf{L}',$$

separating into center-of-mass and “internal motions” $\mathbf{L}'$. 

3
7. The velocity relation for \( \mathbf{r} = \mathbf{R} + \mathbf{s} \) is

\[
\frac{d\mathbf{r}}{dt}{|_{\text{inertial}}} = \dot{\mathbf{R}} + \frac{d\mathbf{s}}{dt}{|_{\text{inertial}}}
\]

\[\Rightarrow \frac{d\mathbf{s}}{dt}{|_{\text{cm}}} = \frac{d\mathbf{s}}{dt}{|_{\text{inertial}}} \quad \text{(for use in } L')\]

\[
L' = \sum_p m_p \mathbf{s}_p \times \frac{d\mathbf{s}_p}{dt}{|_{\text{cm}}}
\]

8. The time derivatives are, with \( \mathbf{F}^{(e)} = \sum_p \mathbf{F}^{(e)}_p \) the total applied force:

\[
\frac{dL_{\text{cm}}}{dt}{|_{\text{inertial}}} = \mathbf{R} \times \mathbf{F}^{(e)}
\]

\[
\frac{dL'}{dt}{|_{\text{inertial}}} = \sum_p \mathbf{s}_p \times \mathbf{F}^{(e)}_p
\]

\[
\frac{dL'}{dt}{|_{\text{cm}}} = \frac{dL'}{dt}{|_{\text{inertial}}}
\]

9. Summary: the torque equation

\[
\frac{d\mathbf{L}}{dt}{|_{\text{inertial}}} = \mathbf{\Gamma}^{(e)}
\]

is valid for the total angular momentum as seen in any inertial frame and relative to instantaneous center of mass. (FW Sec. 27)

(a) When the origin of coordinates is fixed in some inertial frame, this is the equation of motion for angular momentum and torque calculated relative to that point.

(b) This also remains valid when the origin is at the center of mass, even though the c.m. may be in accelerated translation. (P5.1)