Lecture 22: Rigid body rotation II (22 Oct 14)

A. Review: equation of motion

1. For the position \( \mathbf{r} \), the relation between inertial and body frame time derivatives is
   \[
   \mathbf{v} = \left. \frac{d\mathbf{r}}{dt} \right|_{\text{inertial}} = \left. \frac{d\mathbf{r}}{dt} \right|_{\text{body}} + \vec{\omega} \times \mathbf{r}
   \]

2. For fixed \( \mathbf{r} \) in the body frame coordinates; the rotational kinetic energy and angular momentum are
   \[
   T = \frac{1}{2} \sum_p m_p \mathbf{v}_p^2 = \frac{1}{2} \sum_p m_p (\vec{\omega} \times \mathbf{r}_p) \cdot (\vec{\omega} \times \mathbf{r}_p) = \frac{1}{2} \vec{\omega} \cdot \mathbf{I} \cdot \vec{\omega}
   \]
   \[
   \mathbf{L} = \mathbf{I} \cdot \vec{\omega}; T = \frac{1}{2} \vec{\omega} \cdot \mathbf{L}
   \]

3. Now allow for net translational motion and transform to a body-fixed set of axes in c.m. frame. Define \( \mathbf{R} = \) center of mass position and \( \mathbf{s} = \) position relative to the c.m., \( \mathbf{r} = \mathbf{R} + \mathbf{s} \), with
   \[
   \sum_p m_p \mathbf{s}_p = 0
   \]
   \[
   \frac{d\mathbf{r}}{dt} \bigg|_{\text{inertial}} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}}{dt} \bigg|_{\text{inertial}} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}}{dt} \bigg|_{\text{body}} + \vec{\omega} \times \mathbf{s}
   \]

4. The total kinetic energy is a sum of cm kinetic energy and internal \( T' \).
   \[
   T = \frac{1}{2} \sum_p m_p (\dot{\mathbf{R}} + \dot{\mathbf{s}}_p)^2 = \frac{1}{2} \dot{\mathbf{R}}^2 \sum_p m_p + \frac{1}{2} \sum_p m_p \dot{\mathbf{s}}_p^2 \equiv \frac{M}{2} \dot{\mathbf{R}}^2 + T'
   \]

5. The total angular momentum is ("cm + internal \( \mathbf{L}' \)"
   \[
   \mathbf{L} = \sum_p m_p (\mathbf{R} + \mathbf{s}_p) \times \left( \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{s}_p}{dt} \bigg|_{\text{inertial}} \right) = M \mathbf{R} \times \frac{d\mathbf{R}}{dt} + \sum_p m_p \mathbf{s}_p \times \frac{d\mathbf{s}_p}{dt} \bigg|_{\text{inertial}}
   \]

6. The time derivatives are, with \( \mathbf{F}^{(e)} = \sum_p \mathbf{F}_p^{(e)} \) the total applied force:
   \[
   \left. \frac{d\mathbf{L}}{dt} \right|_{\text{inertial}} = \mathbf{R} \times \mathbf{F}^{(e)}; \quad \left. \frac{d\mathbf{L}'}{dt} \right|_{\text{inertial}} = \sum_p \mathbf{s}_p \times \mathbf{F}_p^{(e)}
   \]
B. Euler’s equations

1. The time derivative equations give the torque equation

\[ \frac{d\mathbf{L}}{dt}|_{\text{inertial}} = \mathbf{\Gamma}(e), \]

which is valid (a) for the total angular momentum as seen in any inertial frame and (b) relative to instantaneous center of mass.

(a) When the origin of coordinates is fixed in some inertial frame, this is the equation of motion for angular momentum and torque calculated relative to that point.

(b) It is also valid when the origin is at the center of mass, even though the c.m. may be in accelerated translation. (P5.1)

2. Hence the equation of motion, in terms of torques, for the angular momentum relative to the c.m., [notation \( L' \rightarrow L \)], is

\[ \frac{d\mathbf{L}}{dt}|_{\text{inertial}} = \frac{d\mathbf{L}}{dt}|_{\text{body}} + \mathbf{\omega} \times \mathbf{L} = \mathbf{\Gamma}; \rightarrow \frac{d\mathbf{L}}{dt}|_{\text{body}} = -\mathbf{\omega} \times \mathbf{L} + \mathbf{\Gamma} \]

3. Take the \((t\text{-dependent})\) body-fixed axes to be the principal axes of the moment of inertia tensor (3 orthogonal vectors \( \hat{e}_j \), eigenvalues \( I_j \)),

\[ \mathbf{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3; \ L = I_1 \omega_1 \hat{e}_1 + ...; \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \]

The body frame \( d\mathbf{L}/dt \) arises from the \( t\text{-dependence of the } \omega_j(t). \)

\[ I_1 \frac{d\omega_1}{dt} = \omega_2 \omega_3 (I_2 - I_3) + \Gamma_1 \text{ (and cyclic)} \]

4. Note that in this form the vector components are projected onto body-fixed time-dependent axes. The solution for motion with external torques requires a self-consistent analysis.

C. Compound pendulum

1. Rigid body constrained to rotate about a stationary fixed axis \( \hat{e}_3 \). (i.e., B.1.a condition) FW Sec. 28 –center of percussion]
2. Locate the center-of-mass (cm) in the plane perpendicular to \( \hat{e}_3 \), point Q is intersection of that plane with the axis. Then \( \ell \hat{e}_1 \) is the vector to the cm and \( \hat{e}_2 \equiv \hat{e}_3 \times \hat{e}_1 \).

3. Rotation about the pivot, angle \( \phi \), angular velocity \( \omega_3 = \dot{\phi} \),

\[
\frac{dL_3}{dt} = \Gamma_3; L_3 = I_{33}\omega_3
\]

The axis “3” need not be a principal axis of the tensor \( I \) [see L & L, pp.102-103], but \( I_{33} \) can be evaluated here as

\[
I_{33} = \int d\mathbf{r} \rho(\mathbf{r})[r^2 - r_3^2] \equiv I
\]

4. Torque is the gravitational force acting at the cm (with angle \( \phi \) from the vertical and positive torque taken counter-clockwise):

\[
\mathbf{\Gamma} = M\ell \hat{e}_1 \times \mathbf{g} = -Mg\ell \sin \phi \hat{e}_3
\]

5. The equation of motion, and the small-angle approximation to it, are

\[
I \ddot{\omega} = -Mg\ell \sin \phi \simeq -Mg\ell \phi
\]

so the angular oscillation frequency is \( \Omega = \sqrt{Mg\ell/I} \).

**D. Symmetric top – torque free motion**

1. Solve Euler equations for torque free motion of symmetric top, \([B.1.b]\)

\[
I_1 = I_2 \neq I_3
\]

2. Then \( \omega_3 \equiv \Omega_0 \) is constant and the equations for the 1,2 components are simply coupled:

\[
I_1 \dot{\omega}_1 = \omega_2 \Omega_0 (I_1 - I_3); I_1 \dot{\omega}_2 = -\omega_1 \Omega_0 (I_1 - I_3)
\]

3. Notice that this gives \( d(\omega_1^2 + \omega_2^2)/dt = 0 \) (\( \omega_\perp = \text{constant} \)) and in fact the time dependence is given by

\[
\omega_1(t) = \omega_\perp \cos(\Omega t + \delta); \omega_2(t) = \omega_\perp \sin(\Omega t + \delta); \Omega = \Omega_0 \frac{I_3 - I_1}{I_1}
\]
4. There is a distinction between the direction of the angular momentum vector (constant in the absence of torque) and the angular velocity.

5. This is the phenomenon of precession.

(a) Project $\mathbf{L}$ onto symmetry axis: $\text{const} = \hat{e}_3 \cdot \mathbf{L} = I_3 \omega_3 \equiv |\mathbf{L}| \cos \theta$.

(b) $\vec{\omega}_\perp \equiv \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2$, $\hat{e}_3$, and $\mathbf{L}$ all lie in the same plane $[\mathbf{L} = I_3 \omega_3 \hat{e}_3 + I_1 \vec{\omega}_\perp]$ shows $\mathbf{L}$ as a linear combination of $\vec{\omega}_\perp, \hat{e}_3$. The time dependence of $\hat{e}_3$ in “lab-frame” is

$$d\hat{e}_3/dt = \vec{\omega} \times \hat{e}_3 = -\omega_1 \hat{e}_2 + \omega_2 \hat{e}_1, \quad \perp \hat{e}_3, \mathbf{L}, \vec{\omega}_\perp$$

Hence $d\hat{e}_3/dt$ has magnitude $\omega_\perp$. The precession of the “tip” of $\hat{e}_3$ about the $\mathbf{L}$ axis then has a period $\tau = 2\pi \sin \theta / \omega_\perp$ and the precession frequency is

$$\Omega_{\text{precess}} = \omega_\perp / \sin \theta$$

The angle and frequency are given using [so to LL Sec 33]

$$\cos \theta = I_3 \omega_3 / |\mathbf{L}|; \sin^2 \theta = 1 - \cos^2 \theta = (I_1 \omega_\perp / |\mathbf{L}|)^2$$

$$\Omega_{\text{precess}} = |\mathbf{L}| / I_1$$

E. Asymmetric top – torque free motion

1. 3 distinct moments of inertia.

2. Two constants of motion $\mathbf{L}^2$ and the rotational kinetic energy.

3. Look at small perturbations $\eta_j$ to motion $\vec{\omega} = (\omega_0 + \eta_3) \hat{e}_3 + \eta_1 \hat{e}_1 + \eta_2 \hat{e}_2$ (arbitrarily chosen 3-axis).

$$I_1 d\eta_1 / dt \simeq \omega_0 \eta_2 (I_2 - I_3); I_2 d\eta_2 / dt \simeq \omega_0 \eta_1 (I_3 - I_1); d\eta_3 / dt \simeq 0$$

$$I_1 I_2 d^2 \eta_1 / dt^2 \simeq \omega_0^2 (I_2 - I_3)(I_3 - I_1) \eta_1$$

4. Stable against perturbation (i.e., oscillatory) for $(I_2 - I_3)(I_3 - I_1) < 0$, $I_3$ either the largest or the smallest of the 3 moments of inertia. Otherwise, exponentially growing, unstable perturbation.