Lecture 24: Euler angles (27 Oct 14)

A. Earth as a symmetric top, G5.6,5.8

1. Use datum from P5.7: earth as a symmetric top \( \alpha \equiv (I_3 - I_1)/I_3 = 1/305.3 = 3.28 \times 10^{-3} \) and \( (I_3 = I_1)/I_1 = \alpha/(1 - \alpha) = 3.29 \times 10^{-3} \).

2. Then the precession frequency relative to earth’s rotation frequency is

\[
\Omega = \omega_3^0 (I_3 - I_1)/I_1
\]

and the period for \( \Omega \) is 1 day/\((0.00329) = 304 \) days, approximately 10 months.

3. There is a torque on the non-spherical earth from the sun and moon. Let the sun or moon be a mass point \( M \) and calculate the gravitational energy of the earth as in P5.7, rotation/symmetry axis \( \hat{z} \):

\[
\Phi = -GM \int ds \frac{\rho(s)}{|R - s|} \simeq -GMm_e R + \frac{GM}{R^3} (I_3 - I_1)P_2(\hat{R} \cdot \hat{z})
\]

4. For the calculation in E, will need the gravitational potential averaged over many rotation periods. Let the projection of the \( z \)-axis onto the plane of orbit \( \hat{R} \) be \( \sin \theta \cos \phi \) (spherical polars) and average \( \cos^2 \phi \rightarrow 1/2 \). The \( \theta \)-dependent potential that will be used in the following is

\[
V_2(\cos \theta) = -\frac{GM(I_3 - I_1)}{2R^3}P_2(\cos \theta).
\]

B. Rotations as orthogonal transformations

1. Generalization: a rotation is an orthogonal transformation preserves the scalar product \((a, b)\) for arbitrary vectors \( a, b \):

\[
a_f = M \cdot a; b_f = M \cdot b
\]

\[
(a_f, b_f) \equiv a_f^T \cdot b_f = a^T \cdot M^T M \cdot b \Rightarrow M^T M = I
\]

2. Prove \( MM^T = I \) by choice \( a = M^T \cdot a_f \); then \( a^T \cdot b = a_f^T \cdot M \cdot M^T \cdot b_f \)
C. Euler angles

1. FW sec 29. Start from a set of space-fixed (inertial) Cartesian axes \((\hat{e}_0^0, \hat{e}_0^1, \hat{e}_0^3)\)

2. Three successive rotations to get to the principal axes of the moment of inertia tensor: (1) angle \(\alpha\) around the original \(\hat{z} = \hat{e}_0^3\); (2) angle \(\beta\) around the new “y” = “2” axis; (3) angle \(\gamma\) around the new (and final) \(\hat{z}\) axis \(\hat{e}_3\). [This is the conventional set for QM angular momentum.]

\[
A(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B(\beta) = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 1 & \cos \beta \end{pmatrix}
\]

3. The final result after matrix multiplication for the successive rotations \(C(\gamma)B(\beta)A(\alpha)\) is, using a notation \(C_\alpha = \cos \alpha, S_\alpha = \sin \alpha\), etc.

\[
CBA = \begin{pmatrix} C_\gamma C_\beta C_\alpha - S_\gamma S_\alpha & C_\gamma C_\beta S_\alpha + S_\gamma C_\alpha & -C_\gamma S_\beta \\ -S_\gamma C_\beta C_\alpha - C_\gamma S_\alpha & -S_\gamma C_\beta S_\alpha + C_\gamma C_\alpha & S_\gamma S_\beta \\ S_\beta C_\alpha & S_\beta S_\alpha & C_\beta \end{pmatrix}
\]

4. Relation to spherical polars \((\gamma = 0)\): \(BA\) takes a column vector \((x, y, z) = (rC_\alpha S_\beta, rS_\alpha S_\beta, rC_\beta)\) into \((0, 0, r)\)

\[
BA = \begin{pmatrix} C_\beta C_\alpha & C_\beta S_\alpha & -S_\beta \\ -S_\alpha & C_\alpha & 0 \\ S_\beta C_\alpha & S_\beta S_\alpha & C_\beta \end{pmatrix}
\]

5. Relate rotation axes to principal axes (body-fixed):

(a) Axis for \(\gamma\) (= symmetry axis of the symmetric top) is \(\hat{e}_\gamma \equiv \hat{e}_3\).

(b) Axis for \(\alpha\) was the original \(\hat{e}_3^0\), \(\hat{e}_\alpha = \hat{e}_3^0\); express that in terms of the final axes by the transform: (so \(\hat{e}_3^0 \cdot \hat{e}_3 = C_\beta\))

\[
\hat{e}_\alpha = CBA \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -C_\gamma S_\beta \hat{e}_1 + S_\gamma S_\beta \hat{e}_2 + C_\beta \hat{e}_3
\]
(c) \( \dot{e}_\beta \) was the second stage of \( y \)-axis and is at angle \( \gamma \) from the final \( y \), which is \( \dot{e}_2 \)

\[
\dot{e}_\beta = \sin \gamma \hat{e}_1 + \cos \gamma \hat{e}_2 = S_\gamma \hat{e}_1 + C_\gamma \hat{e}_2
\]

6. Orthogonality relations: \( \hat{e}_\alpha \cdot \hat{e}_\beta = 0; \hat{e}_\beta \cdot \hat{e}_\gamma = 0; \hat{e}_\alpha \cdot \hat{e}_\gamma = C_\beta \)

\[
\hat{e}_3 \times \hat{e}_3^{(0)} = \hat{e}_\gamma \times \hat{e}_\alpha = -S_\beta \hat{e}_\beta
\]

(a) The angular velocities in terms of the Euler angles are

\[
\vec{\omega}_\alpha = \dot{\alpha} \hat{e}_\alpha, \quad \vec{\omega}_\beta = \dot{\beta} \hat{e}_\beta, \quad \vec{\omega}_\gamma = \dot{\gamma} \hat{e}_\gamma
\]

and the vector sum is

\[
\vec{\omega} = \vec{\omega}_\alpha + \vec{\omega}_\beta + \vec{\omega}_\gamma = \dot{\alpha} \hat{e}_\alpha + \dot{\beta} \hat{e}_\beta + \dot{\gamma} \hat{e}_\gamma
\]

(b) The components relative to principal axes \( \omega_i = \vec{\omega} \cdot \hat{e}_i \) are

\[
\omega_1 = -\dot{\alpha} S_\beta C_\gamma + \dot{\beta} S_\gamma; \quad \omega_2 = \dot{\alpha} S_\beta S_\gamma + \dot{\beta} C_\gamma; \quad \omega_3 = \dot{\alpha} C_\beta + \dot{\gamma}
\]

7. The kinetic energy term in the Lagrangian is

\[
T = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)
\]

D. Kinetic energy, canonical momenta of symmetric top

1. The kinetic energy of the symmetric top is

\[
T = \frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2) = \frac{I_1}{2}[\dot{\alpha}^2 S_\beta^2 + \dot{\beta}^2] + \frac{I_3}{2}[\dot{\alpha} C_\beta + \dot{\gamma}]^2
\]

2. Define canonical (angular) momenta \( p = \partial L/\partial \dot{q} \), \( L = T \)

\[
p_\alpha = I_1 \dot{\alpha} S_\beta^2 + I_3 C_\beta [\dot{\alpha} C_\beta + \dot{\gamma}]
\]

\[
p_\gamma = I_3 [\dot{\alpha} C_\beta + \dot{\gamma}] = I_3 \omega_3
\]

\[
p_\beta = I_1 \dot{\beta}
\]
3. The $p_\delta$ are the projections of the angular momentum vector $[\delta = \alpha, \beta, \gamma]$

$$\vec{L} \cdot \hat{e}_\delta = (I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3) \cdot \hat{e}_\delta$$

4. If $\alpha$ and $\gamma$ are cyclic variables (only $\dot{\alpha}, \dot{\gamma}$ in $L$) $\Rightarrow$ the $p_\alpha$ and $p_\gamma$ are constants of the motion. This leaves only the Lagrange equation of motion for $\beta$ “to be solved”:

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = \frac{d}{dt} I_1 \dot{\beta} - \frac{\partial L}{\partial \beta}$$

5. This will be easier to solve after the reduction used to construct the Hamiltonian:

$$\dot{p}_\alpha + \beta \dot{p}_\beta + \dot{\gamma} p_\gamma - T = \frac{I_1 \dot{\beta}^2}{2} + \frac{I_1 \dot{\alpha}^2 \sin^2 \beta}{2} + \frac{I_3}{2} (\dot{\alpha} \cos \beta + \dot{\gamma})^2$$

$$= \frac{I_1 \dot{\beta}^2}{2} + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta}$$

6. Quantum mechanics of the free symmetric top: the cross terms in $p_\alpha, p_\gamma$ arise because this is not an “orthogonal curvilinear system” and going to differential operator for kinetic energy requires a result of differential geometry [Schrödinger, 1926].

**E. Symmetric top in an external field**

1. Set this up for the earth using the potential $V_2$ of part A. The polar angle $\theta$ there is the Euler angle $\beta$. The top (earth) is tilted at an angle $\beta$ to the vertical – set by the orbital plane. The potential is $V_2(\cos \beta)$ and the Lagrangian is [spherical polars $\beta = \theta$; $\alpha = \phi$]

$$L = \frac{I_1}{2} [\dot{\alpha}^2 S_\beta^2 + \dot{\beta}^2] + \frac{I_3}{2} [\dot{\alpha} C_\beta + \dot{\gamma}]^2 - V_2(\cos \beta)$$

2. $\alpha, \gamma$ are cyclic coordinates $\Rightarrow$ $p_\alpha$ and $p_\gamma$ are constants of the motion.