Lecture 25: Differential geometry (29 Oct 14)

0. review 27 Oct homework

A. Symmetric top in an external field

1. Principal axes and rotation axes:
\[ \hat{e}_\gamma = \hat{e}_3; \hat{e}_\beta = S_\gamma \hat{e}_1 + C_\gamma \hat{e}_2; \hat{e}_\alpha = S_\beta (-C_\gamma \hat{e}_1 + S_\gamma \hat{e}_2) + C_\beta \hat{e}_3 \]

2. Kinetic energy was \( T = (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)/2 \)
\[ \omega_1 = -\dot{\alpha} S_\beta C_\gamma + \dot{\beta} S_\gamma; \omega_2 = \dot{\alpha} S_\beta S_\gamma + \dot{\beta} C_\gamma; \omega_3 = \dot{\alpha} C_\beta + \dot{\gamma} \]

3. Now set up symmetric top problem for the earth using the potential \( V_2 \) of L24.A. The polar angle \( \theta \) there is the Euler angle \( \beta \). The top (earth) is tilted at angle \( \beta \) to the vertical – set by the orbital plane. The potential is \( V_2(\cos \beta) \) and the Lagrangian is
\[ L = \frac{I_1}{2} [\dot{\alpha}^2 S_\beta^2 + \dot{\beta}^2] + \frac{I_3}{2} [\dot{\alpha} C_\beta + \dot{\gamma}]^2 - V_2(\cos \beta) \]

4. \( \alpha, \gamma \) are cyclic coordinates \( \Rightarrow p_\alpha \) and \( p_\gamma \) are constants of the motion.
\[ p_\beta = I_1 \dot{\beta}; p_\gamma = I_3 (\dot{\alpha} C_\beta + \dot{\gamma}); p_\alpha = I_1 \dot{\alpha} S_\beta + I_3 C_\beta (\dot{\alpha} C_\beta + \dot{\gamma}) \]

5. The Hamiltonian gives a first integral of the Lagrange equation for \( p_\beta \) (conservation of energy). Check
\[ \dot{p}_\alpha + \dot{\beta} p_\beta + \dot{\gamma} p_\gamma = 2T; \dot{\alpha} S_\beta = (p_\alpha - C_\beta p_\gamma)/I_1 S_\beta \]

6. The result is an expression for the energy (Hamiltonian)
\[ H = \frac{1}{2} I_1 \dot{\beta}^2 + V_{\text{eff}}(\beta) \]
\[ V_{\text{eff}}(\beta) = \frac{1}{2 I_1 S_\beta^2} (p_\alpha - C_\beta p_\gamma)^2 + \frac{p_\gamma^2}{2 I_3} + V_2(\cos \beta) \]
7. The equation of motion for $\beta$ is, using Lagrange
\[ I\ddot{\beta} = \partial L / \partial \beta = I_1 \dot{\alpha}^2 S_\beta C_\beta - \dot{\alpha} S_\beta I_3 (\dot{\alpha} C_\beta + \dot{\gamma}) - \partial V_2 / \partial \beta \]
\[ = I_1 \dot{\alpha}^2 S_\beta C_\beta - \dot{\alpha} S_\beta I_3 \omega_3 - \partial V_2 / \partial \beta \]

8. Treat the case of uniform precession $\beta = \text{constant}$:
\[ 0 = I_1 \dot{\alpha}^2 S_\beta C_\beta - \dot{\alpha} S_\beta I_3 \omega_3 - \partial V_2 / \partial \beta \]
and then slow precession $\dot{\alpha} \ll \omega_3$
\[ \dot{\alpha} S_\beta I_3 \omega_3 \simeq -\partial V_2 / \partial \beta \]

9. G p.228 for numbers: the solar induced precession gives 1 rotation in 81,000 years; the moon’s effect is much larger and, with the sun, gives a period of 26,000 years. There is a related effect for satellite orbits that is much larger.

10. Use of effective potential for the “pure” symmetric top, FW Sec. 31.

**B. Hamiltonian of the symmetric top**

1. The kinetic energy of the symmetric top is
\[ T = \frac{I_1}{2} (\omega_1^2 + \omega_2^2) + \frac{I_3}{2} \omega_3^2 = \frac{I_1}{2} [\dot{\alpha}^2 S_\beta^2 + \dot{\beta}^2] + \frac{I_3}{2} [\dot{\alpha} C_\beta + \dot{\gamma}]^2 \]

2. The canonical momenta $p = \partial L / \partial \dot{q}$, $L = T$
\[ p_\alpha = I_1 \dot{\alpha} S_\beta + I_3 C_\beta [\dot{\alpha} C_\beta + \dot{\gamma}]; \quad p_\gamma = I_3 [\dot{\alpha} C_\beta + \dot{\gamma}] = I_3 \omega_3; \quad p_\beta = I_1 \dot{\beta} \]
are the projections of the angular momentum vector $[\delta = \alpha, \beta, \gamma]$
\[ \vec{L} \cdot \hat{e}_\delta = (I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3) \cdot \hat{e}_\delta \]

3. The kinetic energy was given in (A):
\[ T = \frac{I_1 \dot{\beta}^2}{2} + \frac{I_1 \dot{\alpha}^2 \sin^2 \beta}{2} + \frac{I_3}{2} (\dot{\alpha} \cos \beta + \dot{\gamma})^2 = \frac{p_\beta^2}{2I_1} + \frac{p_\alpha^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} \]

4. Quantum mechanics of the free symmetric top: the cross terms in $p_\alpha, p_\gamma$ arise because this is not an “orthogonal curvilinear system” and going to differential operator for kinetic energy requires a result of differential geometry [Schrödinger, 1926].
C. Differential geometry I

1. Change variable names from \( \beta, \alpha, \gamma \) to \( \theta, \phi, \psi \) for match to spherical polars plus body rotation and use \( A = I_1, B = I_3 \) (symmetric top, “3” = symmetry axis). The kinetic energy is

\[
T = \frac{1}{2} \left[ A(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + C(\dot{\phi} \cos \theta + \psi)^2 \right]
\]

2. The kinetic energy is written with a “covariant” tensor \( g_{ik} \) and length-squared in a general curvilinear coordinate system as

\[
T = \frac{1}{2} \sum_{ik} g_{ik} \dot{q}^i \dot{q}^k; \quad ds^2 = \sum_{ik} g_{ik} dq^i dq^k
\]

where the superscript designates vectors as “contravariant”. The corresponding gradients are “covariant” vectors.

3. In terms of the coefficients in \( T \), the tensor \( g_{ik} \) is

\[
g_{ik} = \begin{pmatrix} g_{\theta\theta} & g_{\phi\theta} & g_{\psi\theta} \\ g_{\phi\theta} & g_{\phi\phi} & g_{\phi\psi} \\ g_{\psi\theta} & g_{\psi\phi} & g_{\psi\psi} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & A \sin^2 \theta + C \cos^2 \theta & C \cos \theta \\ 0 & C \cos \theta & C \end{pmatrix}
\]

4. The determinant of \( g_{ik} \) is

\[
g = \det g = A^2 C \sin^2 \theta
\]

and the contravariant tensor \( g^{ik} = G_{ik}/\det g \), \( G_{ik} \) the minor of \( g_{ik} \) in the determinant is the inverse of \( g_{ik} \):

\[
g^{ik} = \begin{pmatrix} g^{\theta\theta} & g^{\phi\theta} & g^{\psi\theta} \\ g^{\phi\theta} & g^{\phi\phi} & g^{\phi\psi} \\ g^{\psi\theta} & g^{\psi\phi} & g^{\psi\psi} \end{pmatrix} = \begin{pmatrix} \frac{1}{A} & 0 & -\frac{\cos \theta}{A \sin^2 \theta} \\ 0 & \frac{1}{A \sin^2 \theta} & \frac{\cos \theta}{A \sin^2 \theta} \\ 0 & -\frac{\cos \theta}{A \sin^2 \theta} & \frac{1}{A} + \frac{\cos \theta}{A \sin^2 \theta} \end{pmatrix}
\]

5. Now the Laplacian has the form:

\[
\nabla^2 = \frac{1}{\sqrt{g}} \sum_{ik} \frac{\partial}{\partial q^i} \sqrt{g} g^{ik} \frac{\partial}{\partial q^k}
\]

6. The kinetic energy operator then is

\[
T = -\frac{\hbar^2}{2A} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} - 2 \frac{\cos \theta}{\sin \theta} \frac{\partial^2}{\partial \phi \partial \psi} + \frac{A}{C} + \frac{\cos^2 \theta}{\sin^2 \theta} \frac{\partial^2}{\partial \psi^2} \right]
\]
D. Differential geometry II

1. Fill in some of the steps in C [3D: textbook presentation in Margenau and Murphy; higher dimension, e.g. relativity, in Pauli Handbuch der Physik review]

2. Superscript denotes vectors; $ds^2 = g_{ij} dq^i dq^j$ with summation convention; line element (no sum) $ds_i = \sqrt{g_{ii}} dq^i$

3. Say we transform from $x^j$ to $q^j$, then the transform of a differential of a vector is

$$dr = \sum_j \frac{\delta r}{\delta q^j} dq^j \equiv \sum_j e_j dq^j.$$  

Additionally, the original volume element transforms as $d\tau = \{dq\} \sqrt{\det g}$. The $e_i$ are covariant vectors (gradients) and writing an arbitrary vector as $A = a^i e_i$, the $a^i$ are contravariant components of $A$. The reciprocal base vectors $e^j$ to the base $e_i$ are

$$e^i = e_j \times e_k / \left[ e^i \cdot (e_j \times e_k) \right] \text{ cyclic}$$

4. The element of (squared) length is now written

$$ds^2 = (dr) \cdot (dr) = \sum_{ij} e_i \cdot e_j dq^i dq^j = \sum_{ij} g_{ij} dq^i dq^j; e^i \cdot e^j = g^{ij}$$

The contravariant tensor $g^{ij}$ is the inverse of $g_{ij}$

5. The sum of gradient squared has

$$\sum_i (\partial F/\partial x^i)^2 = \sum_{ijk} \frac{\partial F}{\partial q^i} \frac{\partial F}{\partial q^j} \frac{\partial q^k}{\partial x^i} \frac{\partial q^k}{\partial x^j} \equiv \sum_{jk} g^{jk} \frac{\partial F}{\partial q^i} \frac{\partial F}{\partial q^j}$$

and the $g^{jk}$ is inverse to

$$g_{ij} = \sum_k \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j}; \sum_j g_{ij} g^{jk} = \sum_a \frac{\partial x^a}{\partial q^i} \frac{\partial x^a}{\partial q^j} = \delta_{ij}$$

6. The expression for the Laplacian follows then by integration by parts of

$$\int d\tau (\nabla F)^2 = - \int d\tau F \nabla^2 F = \int \{dq\} \sqrt{\det g} \sum_{jk} g^{jk} \frac{\partial F}{\partial q^i} \frac{\partial F}{\partial q^j}.$$