Lecture 29: Strings I (7 Nov 14)

Nov 10 HW now due Nov 12

A. Review (Mostly) – continuum string

1. L16, L17; FW Secs. 25 and 38; G Secs. 13.1, 13.2.

2. The equation of motion followed by generalizing Hamilton’s principle to cover the Lagrangian density $L$

$$L = \int dx L(u, \partial u/\partial x, \partial u/\partial t)$$

$$\delta \int_{t_1}^{t_2} dt \int_x^x dx L(u, \partial u/\partial x, \partial u/\partial t) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial u/\partial t)} + \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial u/\partial x)} - \frac{\partial L}{\partial u} = 0$$

3. The Lagrangian density for transverse $(u)$ vibrations of a 1D string with spatially varying mass density $\sigma(x)$ and tension $\tau(x)$ is, calculating the energy of stretching the string by

$$\delta E = \tau(x)[\sqrt{1 + (\partial u/\partial x)^2} - 1]dx \simeq \frac{1}{2}\tau(x)(\partial u/\partial x)^2 dx$$

$$L = \frac{1}{2}\sigma(x)(\frac{\partial u}{\partial t})^2 - \frac{1}{2}\tau(x)(\frac{\partial u}{\partial x})^2$$

4. Hamiltonian formulation G Sec. 13.4 (FW Sec. 45), using notation $\dot{u} = \partial u/\partial t$; Hamiltonian density $H$:

$$\mathcal{P} = \partial L/\partial \dot{u} = \sigma(x)\frac{\partial u}{\partial t}; \quad H = \mathcal{P}\dot{u} - L$$

$$H = \int \mathcal{H} dx = \int dx \left[\frac{1}{2}\sigma(x)(\frac{\partial u}{\partial t})^2 + \frac{1}{2}\tau(x)(\frac{\partial u}{\partial x})^2\right]$$

5. The Euler-Lagrange equations for the string $L$ (in [3]) are

$$\sigma(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\tau(x)\frac{\partial u}{\partial x}$$
6. The case \( \sigma(x) = \sigma \) and \( \tau(x) = \tau \) (constants) gives rise to the elementary form of the wave equation with constant speed \( c^2 = \tau/\sigma \):

\[
\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}
\]

(a) Normal modes: separate time, space variables \( u(x,t) = \rho(x) \cos(\omega t + \delta) \) and define \( k^2 = \omega^2/c^2 \). The equation for \( \rho(x) \) is

\[
\frac{d^2 \rho}{dx^2} + k^2 \rho = 0
\]

(b) Discrete allowed values of \( k \) (and hence of \( \omega \)) are set by boundary conditions on a finite length \( 0 < x < \ell \). Fixed ends give normalized \( \rho_n(x) = \sqrt{2/\ell} \sin k_n x; \ k_n = n\pi/\ell \)

(c) The solution for general \((x,t)\) can then be constructed from the initial position \( u(x,0) \), here using the Fourier sine series.

\[
u(x,t) = \sum_{n=1}^{\infty} \rho_n(x)C_n \cos(\omega_n t + \phi_n) = \sum_n \rho_n(x)\zeta_n(t)
\]

where the amplitudes \( \zeta_n \) are set by the initial conditions, e.g.,

\[
\zeta_n(0) = \int_0^\ell \rho_n(x) u(x,0) \, dx
\]

(d) The Lagrangian is then formed by integrating:

\[
L = \int_0^\ell L \, dx = \frac{1}{2} \sum_n (\sigma \dot{\zeta}_n^2 - \tau k_n^2 \zeta_n^2)
\]

e, i.e., a quadratic form in the amplitudes \( \zeta_n \). With a “quantization volume” \( \ell \), the integral over a continuous variable \( x \) becomes a denumerably infinite sum over the index \( n \) on wavelength/wavenumber. The Hamiltonian is \([\pi_n = \partial L/\partial \dot{\zeta}_n = \sigma \dot{\zeta}_n \]

\[
H = \frac{1}{2} \sum_n (\sigma \dot{\zeta}_n^2 + \tau k_n^2 \zeta_n^2)
\]

(e) The basic 1D wave equation can be solved with other boundary conditions – free ends or periodic boundary conditions. Again, satisfy general initial conditions with series of sines and cosines.
B. D’Alembert solution

1. FW Sec. 39 for the elementary wave equation.

2. By inspection one can see that the functional forms \( f(x - ct) \) and \( g(x + vt) \) for arbitrary differentiable functions \( f \) and \( g \) satisfy the equation.

3. The text shows this is the most general solution. A change of variable \( r = x - ct; s = x + ct \) and \( u(x, t) \equiv U(r, s) \) shows that the function \( U(r, s) = u(x, t) \) satisfies

\[
\frac{\partial}{\partial r} \frac{\partial U}{\partial s} = 0
\]

which is satisfied by \( U = \psi(r) + \phi(s) \).

4. FW. p.213 construct a solution to match initial conditions \( u(x, 0) = f(x) \) and \( \dot{u}(x, 0) = g(x) \) for the infinite string. Then pp.214-215 give the adaptation of the formalism for infinite string to finite length.

C. Eigenfunction expansions

1. FW Sec. 40.

2. The 1D wave problem with position-dependent density and tension is a special case of the more general mathematics of Sturm-Liouville theory. The principal results that get used are the orthogonality of solutions, their use as a basis, and node-counting.

3. Generalize the Lagrangian density to \( \mathcal{L} = \mathcal{T} - \mathcal{V} \) with

\[
\mathcal{T} = \frac{1}{2} \sigma(x) \left( \frac{\partial u}{\partial t} \right)^2; \mathcal{V} = \frac{1}{2} \tau(x) \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} v(x) u^2
\]

4. \( \mathcal{L} \) has no explicit time dependence; the Euler-Lagrange equation is

\[
\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial u/\partial t)} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial u/\partial x)} - \frac{\partial \mathcal{L}}{\partial u} = 0
\]

\[
\Rightarrow \sigma(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \tau(x) \frac{\partial u}{\partial x} - v(x) u
\]
5. Look for normal modes (harmonic time-dependence; separation of variables solutions) in the form \( u(x,t) = \rho(x) \cos(\omega t + \delta) \); \( \rho(x) \) satisfies

\[-\frac{d}{dx} \tau(x) \frac{d\rho}{dx} + v(x) \rho = \omega^2 \sigma(x) \rho\]

6. This is an eigenvalue problem (only selected values \( \omega_n \) “work”). The eigenfunctions \( \rho_n \) are orthogonal with “weight” \( \sigma(x) \) and form a complete set. The orthogonality on a finite line is shown “simply” for fixed ends, free ends, and periodic ends. The completeness comes from the discussion at FW pp. 232–235.

7. Some qualitative features of physical spectra \( \omega^2 \): (i) positive eigenvalues (stability, conditions on \( v(x) \)), (2) spectrum bounded from below (not so simple to prove for quantum mechanics), (3) spectrum not bounded from above (no shortest wavelength, largest wave number for the continuum string).

**Homework for Wednesday 12 Nov**

1. P7.3 Energy of static stretched string as in A.3:

\[ \Delta E = \frac{1}{2} \tau \int (\partial y/\partial x)^2 \, dx \]

2. P7.5 (a–d) follow the changes of variable; at free end require

\[ \tau(x) \partial u/\partial x \to 0, x \to \ell \]

3. P7.9 variational calculation following Sec. 41. In choosing trial functions use the symmetry (even/odd) and the result for spatial 1D that the eigenstates can be ordered by the number of interior nodes.

4. P7.18 reflection/transmission coefficients as in Sec 45; matching of solution at point mass as in Sec. 44.