Lecture 34: Hydrodynamics III (19 Nov 14)

A. Review: density and momentum

1. Consider small volume element fixed in space, the change within, and the flows across the bounding surface.

2. Equation of continuity, mass density $\rho$ and stream velocity $\mathbf{v}$ both treated as functions of $(\mathbf{r}(t), t)$; treatment in terms of transport through a fixed small volume is “Eulerian” description. Follow the fluid element with hydrodynamic derivative $d\rho/dt$.

$$0 = \frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \mathbf{\nabla} \rho + \rho \mathbf{\nabla} \cdot \mathbf{v} \equiv \frac{d\rho}{dt} + \rho \mathbf{\nabla} \cdot \mathbf{v}$$

3. Newton #2: acceleration of a fluid element by pressure gradient and body force $\mathbf{f}$ per unit mass. “Ideal” fluid – no shear forces, no viscosity [“Euler’s equation of non-viscous hydrodynamics”]

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \mathbf{\nabla} \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \mathbf{\nabla} P$$

4. Momentum density is $\rho \mathbf{v}$; the change in $dV$ arising from the net outgoing flux $\mathbf{v} \rho v_i$ through the bounding area is, as in Gauss law:

$$- \int_A d\mathbf{A} \cdot \mathbf{v}(\rho v_i) = - \int_V \sum_j \frac{\partial}{\partial x_j} (v_j \rho v_i) dV$$

5. Define the stress tensor (dimensions of force/area = energy/volume)

$$T_{ij} = P \delta_{ij} + \rho v_i v_j$$

6. Momentum balance: momentum density in the volume, net flux across boundaries, and effects of pressure gradient and body forces. Then the partial differential equation:

$$\frac{\partial (\rho v_i)}{\partial t} + \sum_j \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$
B. Conservation of energy

1. Again start from a volume element $dV$. The energy density arises from the kinetic energy of the moving element and the thermodynamic internal energy $\epsilon$ (per unit mass)

$$\frac{1}{2}\rho v^2 + \rho\epsilon(\rho)$$

where $v$ is the average velocity of the volume element.

2. $\epsilon$ has contributions from random (thermal) motions about the average $v$ and intermolecular force terms. In principle it depends on two thermodynamic coordinates, volume (density) and entropy. Relate $\epsilon$ to the internal energy $U$. For mass $M$ in volume $V$, $\rho = M/V$, $U = M\epsilon$. Hence with $dS = 0$ [for an ideal fluid with no dissipation mechanism (no viscosity or thermal conductivity), changes are isentropic]:

$$dU = TdS - PdV \rightarrow d\epsilon/\partial\rho = \partial\epsilon/\partial\rho|_S = P/\rho^2$$

3. Conservation of energy (ideal fluid): the change in the energy of $dV$ arises from the divergence of the current $\mathbf{v}$ across the bounding surface and the work done by pressure and body forces $\mathbf{f}$.

4. Identity and transform of the kinetic energy piece:

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \frac{1}{2}v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) \Rightarrow \mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{v} \cdot \nabla v^2/2$$

$$\frac{\partial}{\partial t} \frac{1}{2} \rho v^2 = \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{2} v^2 \nabla \cdot \rho \mathbf{v} + \rho \mathbf{v} \cdot (\mathbf{f} - \frac{1}{\rho} \nabla P - \nabla \frac{1}{2} v^2)$$

5. Derivatives of $\rho\epsilon$ and $\epsilon = \epsilon(\rho)$, where $h = \epsilon + (P/\rho)$ is the thermodynamic enthalpy per unit mass, $H = U + PV$, $h = H/M$:

$$\frac{\partial \rho\epsilon}{\partial t} = \frac{d\rho\epsilon}{d\rho} \frac{\partial \rho}{\partial t} = (\epsilon + \frac{P}{\rho}) \frac{\partial \rho}{\partial t} \cdot \nabla \epsilon + \frac{P}{\rho} = \frac{1}{\rho} \nabla P$$

6. Collect all the pieces and express the change in the energy of the volume element is terms of the divergence of an energy current $\mathbf{j}_e$ across the bounding surface and the work done by applied body forces $\mathbf{f}$:

$$\frac{\partial}{\partial t}[\frac{1}{2} \rho v^2 + \rho\epsilon] + \nabla \cdot \mathbf{j}_e = \rho \mathbf{f} \cdot \mathbf{v}; \quad \mathbf{j}_e = (\frac{1}{2} \rho v^2 + \rho\epsilon + P)\mathbf{v}$$
C. Bernoulli’s theorem

1. As in (B), isentropic changes. Assume irrotational flow with a velocity potential $\mathbf{v} = -\nabla \Phi$ and a conservative external force $\mathbf{f} = -\nabla U$.

\[
\nabla \times \mathbf{v} = 0 \Rightarrow (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \frac{1}{2} v^2
\]

\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \frac{1}{2} v^2 = -\nabla U - \frac{1}{\rho} \nabla P
\]

\[
-\frac{\partial \nabla \Phi}{\partial t} + \nabla \frac{1}{2} v^2 = -\nabla U - \nabla (\epsilon + \frac{P}{\rho})
\]

2. The last shows the vector Euler equation for $\frac{\partial \mathbf{v}}{\partial t}$ becomes a scalar equation for isentropic and irrotational flow:

\[
\epsilon + \frac{P}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = \text{constant}
\]

(The possible $t$-dependent, spatially constant term on the right-hand side can be removed by a “gauge transformation.” FW pg.301) In the special case of steady flow, it expresses energy conservation.

D. Small amplitude sound waves

1. Conservation laws:

\[
0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}; \quad \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla P
\]

2. The linearized conservation laws are ($\mathbf{v}, \delta \rho, \delta P$ first order in a small parameter)

\[
\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0; \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla \delta P
\]

3. Hence, the constraints/relations for the perturbations are:

(a) Irrotational velocity, $\mathbf{v} = -\nabla \Phi$.

(b) Pressure from second equation $\delta P = \rho_0 \delta \Phi / \partial t$
(c) Density from $\delta P = (dP/d\rho) \delta \rho = c^2 \delta \rho \rightarrow c^2 \delta \rho = \rho_0 \partial \Phi / \partial t$

4. The wave equation then is given in terms of $\Phi$
$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

and boundary conditions on the solution (setting eigenfrequencies, normal modes) come from conditions such as perpendicular velocity at a rigid boundary equals zero: $\hat{u} \cdot \nabla \Phi = 0$. FW pg. 307 for discussion of free surface (open end to a resonator ...).

E. Lagrangian formulation – 2 fields

1. Assume irrotational flow. Now there are two field variables $\rho$ and $\Phi$. Two “equations of motion” = equation of continuity plus Bernoulli,
$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla \Phi) = 0; \quad \epsilon + \frac{P}{\rho} + U + \frac{1}{2} (\nabla \Phi)^2 - \frac{\partial \Phi}{\partial t} = 0$$

2. The Lagrangian density is (checked by getting the equations of motion):
$$\mathcal{L} = \rho \frac{\partial \Phi}{\partial t} - \frac{1}{2} \rho (\nabla \Phi)^2 - \rho \epsilon - \rho U$$
$$0 = \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \Phi} \right) + \nabla \cdot \left( \rho \frac{\partial \mathcal{L}}{\partial \nabla \Phi} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \nabla \Phi)$$
$$0 = - \frac{\partial \mathcal{L}}{\partial \rho} = \frac{1}{2} (\nabla \Phi)^2 - \frac{\partial \Phi}{\partial t} + U + \frac{\partial \rho \epsilon}{\partial \rho}$$

3. The canonical momentum densities and Hamiltonian density are
$$\mathcal{P}_\Phi = \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial t)} = \rho; \quad \mathcal{P}_\rho = \frac{\partial \mathcal{L}}{\partial (\partial \rho / \partial t)} = 0$$
$$\mathcal{H} = \mathcal{P}_\Phi \frac{\partial \Phi}{\partial t} + \mathcal{P}_\rho (\partial \rho / \partial t) - \mathcal{L} = \frac{1}{2} (\nabla \Phi)^2 + \rho U + \rho \epsilon$$

4. Generalize the current calculation for the string to 2 fields $(u_j = \rho, \Phi)$
$$S_i = \sum_j \frac{\partial \mathcal{L}}{\partial (\partial u_j / \partial x_i)} \frac{\partial u_j}{\partial t} = \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial x_i)} \frac{\partial \Phi}{\partial t} + \frac{\partial \mathcal{L}}{\partial (\partial \rho / \partial x_i)} \frac{\partial \rho}{\partial t}$$
$$= - \rho \nabla_i \Phi \frac{\partial \Phi}{\partial t} = v_i [\rho \epsilon + P + \rho U + \frac{1}{2} \rho v^2]$$
as obtained directly using conservation of energy.