Lecture 37: Four-vectors (26 Nov 14)

Homework for December 8 is available as Suppl14.pdf

0. 24 November homework

A. Review: Lorentz transformation

1. Frames $S$ and $S^*$ moving with relative velocity $v$ along $\hat{x}$

   \[ ct^* = \gamma(ct - \beta x); \quad x^* = \gamma(x - \beta ct) \]

   \[ y^* = y; \quad z^* = z; \quad \beta = v/c; \quad \gamma = 1/\sqrt{1 - \beta^2} \]

2. notation for 4-vector: $x^0, \mu = 0, 1, 2, 3$; $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$

3. Velocity transform, “elementary method” with differentials $dx$ and $dt$

   \[ u_x = \frac{u_x^* + v}{1 + (vu_x^*/c^2)}; \quad u_y = \frac{u_y^*}{\gamma [1 + (vu_x^*/c^2)]} \]


   (a) Compare time interval for clock at rest (same position in $S^*$) and when observed in lab frame $S$ (2 places): time dilation.

   \[ t_2 - t_1 = \gamma([t_2^* - t_1^*] + (v/c^2)[x_2^* - x_1^*]) \rightarrow \Delta t = \gamma \Delta t^* \]

   “High energy muons live longer.” Gedanken experiment: measure time by the distance traveled divided by speed of light. Bounce light from a mirror with path at right angles to the relative velocity. Then calculate distance traveled as viewed in rest frame and as viewed in “lab”

   (b) Compare length of rod $L_0$ in rest frame $S^*$ and $L_S$ when measured (same times) in lab frame.[FitzGerald, Lorentz 1892-1893]

   \[ x_2^* - x_1^* = \gamma([x_2 - x_1] - v[t_2 - t_1]) \rightarrow L_0 = \gamma L_S \]

   (c) Relativity of simultaneity: events simultaneous (equal times) in one frame appear at different times in a moving frame – same calculation as in (a).
B. Doppler effect

1. Transformation for plane electromagnetic waves that satisfy the wave equation [the wave operator retains its form under Lorentz transformation to moving frame $S^*$]

$$[\nabla^2 - \frac{\partial^2}{\partial t^2}] u(\mathbf{r}, t) = 0$$

2. Symon, Sec. 13-6; Jackson Sec.11.3.D. Plane e-m waves with forms in $S$ and $S^*$ (the latter being the rest frame of the source).

$$A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta); \quad A^* \cos(\mathbf{k}^* \cdot \mathbf{r}^* - \omega^* t^* + \theta^*)$$

Argue that the phase should be invariant ("counting of crests is independent of frame," Jackson Sec. 11.2.A, Møller, Sec.3). Then plug the expressions for $x^*$, etc. and collect coefficients of $x, t, ...$:

$$k_x = \gamma [k^*_x + (\beta/c)\omega^*]; \quad \omega = \gamma [\omega^* + c\beta k^*_x]$$

$$k_y = k^*_y; \quad k_z = k^*_z$$

3. Notice an invariance under Lorentz transformation:

$$c^2 k^2 - \omega^2 = c^2 k^*^2 - \omega^*^2$$

4. Let propagation in the rest frame be in $x^*-y^*$ plane at angle $\alpha^*$ to $\hat{x}^*$,

$$k^*_x = (\omega^*/c) \cos \alpha^*; \quad k^*_y = (\omega^*/c) \sin \alpha^*$$

Then the Doppler effect is ($\omega$ in lab, higher frequency when approaching, $\alpha^* < \pi/2$)

$$\omega = \gamma \omega^*[1 + \beta \cos \alpha^*]$$

The limit $\gamma \to 1$ gives the classical Doppler effect; there is an additional relativistic effect [clock rate] even if the relative motion is at right angles to the line of sight.

5. Transform of angles (second form was the stellar aberration result)

$$\cos \alpha = k_x c/\omega = \frac{\cos \alpha^* + \beta}{1 + \beta \cos \alpha^*}; \quad \tan(\alpha/2) = \sqrt{1 - \frac{\beta}{1 + \beta}} \tan(\alpha^*/2)$$
6. Non-relativistic Doppler shift by the invariance of the phase of a plane wave under Galilean transformation \( t = t'; r = r' + vt, \ k = k\hat{n}, \ \hat{n} \) is a unit vector, of length 1 in both frames, \( \hat{n} \cdot \hat{n} = \hat{n}' \cdot \hat{n}' = 1 \)

\[
\phi = \omega t - k \cdot r = \omega \left[ t - \frac{\hat{n} \cdot r}{c} \right] = \omega' \left[ t' - \frac{\hat{n}' \cdot r'}{c} \right]
\]

Substitute \( t = t', \ r = r' + vt' \) and equate coefficients:

\[
\omega' = \omega \left[ 1 - \frac{\hat{n} \cdot v}{c} \right]; \ \frac{\omega'}{c} \hat{n}' = \frac{\omega}{c} \hat{n}
\]

\[
\frac{\omega'}{c} = \frac{\omega}{c} \Rightarrow \frac{c'}{c} = \frac{\omega'}{\omega} = \left[ 1 - \frac{\hat{n} \cdot v}{c} \right]; \ \hat{n} = \hat{n}'
\]

C. Space-time algebra

1. Symon Sec. 14-1 [an overall sign difference from Goldstein]

2. Space-time distance between events \( E_1 = (r_1, ct_1) \) and \( E_2 = (r_2, ct_2) \) is

\[
S_{21} = (r_1 - r_2)^2 - c^2(t_1 - t_2)^2 = \sum_{\mu=0}^{3} g_\mu(x_{\mu 1} - x_{\mu 2})^2
\]

with \( g_0 = -1, g_1 = g_2 = g_3 \) [other conventions use overall minus sign \( g_0 = 1, g_1 = g_2 = g_3 = -1 \)]

3. Represent the Lorentz transformation as a matrix operation:

\[
x^*_\mu = \sum_\nu a_{\mu \nu} x_\nu
\]

Composite of two transformations is the matrix product (group property checked for parallel velocities, Poincaré).

4. The Lorentz transformation on \( x_\mu \) preserves the space-time norm:

\[
\sum_\mu g_\mu x^*_\mu x_\mu = \sum_\mu g_\mu \sum_{\nu \lambda} a_{\mu \nu} x_\nu a_{\mu \lambda} x_\lambda = \sum_\nu g_\nu x_\nu x_\nu
\]

\[
\Rightarrow \sum_\mu g_\mu a_{\mu \nu} a_{\mu \lambda} = g_\nu \delta_{\nu \lambda}
\]

16 equations reduce to \( 4 + 6 = 10 \) distinct equations (i.e., 10 constraints – 6 free parameters in \( a_{\mu \nu} \)).
5. Then one can check that the inverse transform is
\[ a^{-1}_{\nu\mu} = g_{\nu\gamma}a_{\gamma\lambda}g^{\lambda}_{\mu\theta} = \sum_{\lambda} g^{\lambda}_{\gamma\mu} \delta_{\nu\lambda} = \delta_{\nu\lambda} \]

6. The defining property of a 4-vector is that its transform is
\[ A^{\ast}_{\mu} = \sum_{\nu} a_{\mu\nu}A_{\nu} \]
Sums of 4-vectors are 4-vectors, multiply by a 4-scalar → a 4-vector.

7. If \( A_{\mu}, B_{\mu} \) are 4-vectors then the “inner product” \( (A_{\mu}, B_{\mu}) = \sum_{\mu} g_{\mu\nu}A_{\mu}B_{\mu} \) is a 4-scalar (invariant). However the sign is not guaranteed.

8. The calculation of the Doppler effect used the requirement that \( k \cdot r - \omega t \) is a Lorentz scalar = the inner product of the 4-vectors \( k, \omega/c \) and \( r, ct \).

9. Derivative calculus: start from
\[ x^{\ast}_{\mu} = \sum_{\nu} a_{\mu\nu}x_{\nu}; x_{\nu} = \sum_{\mu} a^{-1}_{\nu\mu}x^{\ast}_{\mu} \]
and use chain rule derivatives:
\[ \frac{\partial}{\partial x^{\ast}_{\mu}} = \sum_{\nu} \frac{\partial}{\partial x_{\nu}} \frac{\partial x_{\nu}}{\partial x^{\ast}_{\mu}} = \sum_{\nu} \frac{\partial}{\partial x_{\nu}} a^{-1}_{\nu\mu} = \sum_{\nu} g_{\nu\mu}a_{\mu\nu} \frac{\partial}{\partial x_{\nu}} \]
and so to a 4-vector and a 4-scalar (the wave operator)
\[ \Box_{\mu} = g_{\mu\nu} \frac{\partial}{\partial x_{\nu}}; g_{\mu\nu} \frac{\partial}{\partial x_{\nu}} = \sum_{\nu} a_{\mu\nu}g_{\nu\theta} \frac{\partial}{\partial x_{\nu}}; (\Box_{\mu}, \Box_{\mu}) = \sum_{\mu} g_{\mu\nu} \Box_{\mu} = \nabla^{2} - \frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \]

10. Summary: a 4-vector transforms as \( A^{\ast}_{\mu} = \sum_{\nu} a_{\mu\nu}A_{\nu} \) Some examples \( x_{\mu} = (ct, r), k_{\mu} = (\omega/c, k), j_{\mu} = (cp, j), \Box_{\mu} = g_{\mu\nu} \Box_{\nu}, p_{\mu} = (E/c, p) \).
The scalar product of two 4-vectors is a Lorentz invariant:
\[ (A_{\mu}, B_{\mu}) \equiv \sum_{\mu} g_{\mu\nu}A_{\mu}B_{\mu} \]
and \( (\Box_{\mu}, \Box_{\mu}) \) is the wave equation operator; \( (\Box_{\mu}, j_{\mu}) = 0 \) is the equation of continuity.
\[ (\Box_{\mu}, \Box_{\mu}) = \sum_{\mu} g_{\mu\nu} \Box_{\mu} = \nabla^{2} - \frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \]