Physics 715 (Spring 2002)
Problem Set 7

Prob. 1. The mean field approximation for the Ising model yields the result that the mean magnetization $\sigma = \langle s \rangle$ satisfies the equation

$$\sigma = \tanh \left( \frac{T_c}{\sigma} \right),$$

(1)

where $T_c = zJ/2k_B$, with $J$ the strength of the coupling between nearest neighbor lattice sites and $z$ the number of nearest neighbors.

a) Show that $\sigma$ has the following temperature dependence:

(i) $\sigma \approx 1 - 2e^{-2T_c/T}$ if $T \ll T_c$, and (ii) $\sigma \approx \sqrt{3(1 - T/T_c)}$ if $T \approx T_c$.

b) Compute the jump in the heat capacity at $T = T_c$.

c) Compute the magnetic susceptibility, $\chi_{T,N}(B = 0)$, in the neighborhood of $T = T_c$ for both $T > T_c$ and $T < T_c$. What is the critical exponent $\gamma$? Is it the same for $T > T_c$ and $T < T_c$?

a) (i) For $T \ll T_c$, we expect $\sigma \approx \pm 1$. Suppose $\sigma \approx 1$, then:

$$\sigma = \frac{1 - e^{-2\sigma T_c/T}}{1 + e^{-2\sigma T_c/T}} \approx 1 - 2e^{-2\sigma T_c/T} \approx 1 - 2e^{-2T_c/T}$$

(2)

(ii) For $T \approx T_c$, $\sigma \approx 0$; this justifies a Taylor series of the l.h.s.:

$$\sigma \approx \frac{T_c}{T} - \frac{1}{3} \left( \frac{T_c}{T} \right)^3 + \ldots$$

so either $\sigma = 0$ or

$$\sigma \approx \sqrt{3 \left( 1 - \frac{T}{T_c} \right)}.$$  

(3)

b) Differentiating the equation (2) gives

$$\frac{d\sigma}{dT} = \left( 1 - \tanh^2 \left( \frac{T_c}{\sigma} \right) \right) \left( -\frac{T_c}{T^2} \sigma + \frac{T_c}{T} \frac{d\sigma}{dT} \right),$$

so

$$\frac{d\sigma}{dT} = \frac{(1 - \sigma^2) T_c}{(1 - \sigma^2) T^2} - \sigma.$$  

Using this in the expression of the energy

$$\langle E \rangle = -\frac{1}{2} zNJ\sigma^2$$

1
the heat capacity is be found to be

\[ C = \frac{d\langle E \rangle}{dT} = -2Nk_B \frac{\sigma^2(1 - \sigma^2)}{(1 - \sigma^2) \frac{T_c}{T} - 1} \left( \frac{T_c}{T} \right)^2 \]

which gives for \( T \to T_c^- \) (hence \( \sigma \to 0 \)), using the approximation (4),

\[ C \to 2Nk_B \frac{\sigma^2}{1 - \frac{T_c}{T} + \sigma^2} \to 3Nk_B . \quad (4) \]

This is also the jump of \( C \) since \( C = 0 \) for \( T > T_c \).

c) In nonzero magnetic field \( B \), (2) reads

\[ \sigma = \tanh \left( T^{-1}[T_c \sigma + \mu B] \right) \quad [ \k_B \equiv 1 ] , \quad (5) \]

giving the implicit equation for the magnetic susceptibility \( \chi \) at zero field

\[ \chi = \mu \frac{\partial \sigma}{\partial B} = \frac{T^{-1}(T_c \chi + \mu^2)}{\cosh^2(T^{-1}T_c \sigma)} ; \]

yielding

\[ \chi = \frac{T^{-1} \mu^2}{\cosh^2(T^{-1}T_c \sigma) - T^{-1}T_c} . \]

Then using (4),

\[ \chi \approx \frac{\mu^2}{T_c - T} , \quad \text{as } T \to T_c^- \quad (6) \]

\[ \chi \approx \frac{\mu^2}{T - T_c} , \quad \text{as } T \to T_c^+ . \quad (7) \]

So \( \gamma = 1 \) on both sides of \( T_c \).

Prob. 2. Huang, problem 17.3. Consider mean-field theory with a cubic term in the Landau free energy:

\[ \mathcal{F}_L(m) = \frac{1}{2} r m^2 + s m^3 + u m^4 . \]

Assume that \( s \) and \( u > 0 \) are fixed, while \( r \) varies (say, by changing temperature). Sketch the shape of \( \mathcal{F}_L(m) \) for various values of \( r \) to show how the lowest minimum \( \bar{m} \) of \( \mathcal{F}_L(m) \) depends on \( r \). Show that there is a first-order phase transition at a certain value of \( r \). Find that value, and the discontinuity of \( \bar{m} \) at the transition.

The minimum value of the ‘Landau free energy’ \( \mathcal{F}_L \) with respect to the order parameter \( m \) is the (ordinary) free energy of the model. A first order
phase transition occurs when two local minima of $\mathcal{F}_L$, at different values of $m$, have the same depth so that as parameters of $\mathcal{F}_L$ are changed, the global (lowest) minimum of $\mathcal{F}_L$ switches from one local minimum to the other.

If $r > 0$, one sees that the given $\mathcal{F}_L(m)$ has a local minimum of value 0 at $m = 0$, that is,

$$
\mathcal{F}_L(0) = 0 ; \quad \mathcal{F}'_L(0) = 0 ; \quad \mathcal{F}''_L(0) > 0 .
$$

It will have a second minimum also of value 0 at an $m$ ($m \neq 0$) such that $\mathcal{F}_L(m) = 0 , \mathcal{F}'_L(m) = 0 , \text{ and } \mathcal{F}''_L(m) > 0$. The first two of these conditions read

$$
r + 2sm + 2um^2 = 0 , \quad r + 3sm + 4um^2 = 0 .
$$

These simultaneous quadratic equations become two linear equations by eliminating either the constant term (subtract the two equations) or the quadratic term (subtract the two equations, after multiplying the first by 2), giving

$$
m = -s/2u \quad \text{,} \quad m = -r/s \quad \text{, hence (upon eliminating } m) , \quad r = s^2/2u .
$$

The third condition, $\mathcal{F}''_L(m) > 0$, then reads $s^2/2u > 0$ — which is true because $u > 0$.

So there is a phase transition when $r = s^2/2u$, and the jump (discontinuity) of $\tilde{m}$, the value of $m$ at the global minimum, is $\pm s/2u$.

Prob. 3. The Landau free energy for a hypothetical system is of the form

$$
F(t, m, h) = -hm + \frac{1}{4}atm^4 + \frac{1}{6}bm^6 , \quad a, b > 0,
$$

where $m$ is the order parameter, $t = (T - T_c)/T_c$, and the external field $h$ is the variable conjugate to $m$. Show that there is a continuous phase transition at $T = T_c$ for $h = 0$. Determine the critical exponents $\alpha, \beta, \gamma$, and $\delta$, and show directly that

$$
\alpha + 2\beta + \gamma = 2 , \quad \delta = 1 + \frac{\gamma}{\beta} .
$$

Show explicitly that your solution for $m$ actually minimizes $F$ for $t > 0$ and $t < 0$. Keep only the leading behavior in $t$ for $t \to 0$. [Hint: Calculate the thermodynamic derivatives that determine $S$ and $\chi = (\partial h/\partial m)_t^{-1}$, and evaluate them for the value of $m$ that minimizes $F$, and then use $C = T \partial S/\partial T$. To determine $\delta$, minimize $F$ on the critical isotherm $t = 0$ for $h \neq 0$]

$$
F' = -h + (at + bm^2)m^3 ; \quad a, b > 0
$$
For $h = 0$: For $t > 0$, the triple root of $F'$ at $m = 0$ is the minimum of $F$ with value $0$ (its other two roots are complex). For $t < 0$, the triple root of $F'$ at $m = 0$ is a maximum of $F$, while the other two roots at $m = \pm \sqrt{-at/b}$ are minima of $F$ with value $F' = \frac{5}{12}(at)^3/b^2$, $< 0$. So there is a phase transition at $t = 0$ because the equilibrium order parameter $\tilde{m}$ changes from 0 when $t > 0$ to $\neq 0$ when $t < 0$. The phase transition is continuous, because $\tilde{m}$ is continuous (does not jump) at $t = 0$. 

$\alpha$ is defined by $C \sim t^{-\alpha}$. We have

$$C = T \frac{\partial S}{\partial T} = -T \frac{\partial^2 F/\partial T^2}{|h=0|} = -TT_c^{-2} \frac{5}{2} a^3 b^{-2} t , \text{ so } \alpha = -1 .$$

$\beta$ is defined by $\tilde{m} \sim t^{-\beta}$. We have

$$\tilde{m} = \pm \sqrt{-at/b} , \text{ so } \beta = \frac{1}{2} .$$

$\gamma$ is defined by $\chi \sim t^{-\gamma}$. In the disordered phase $\chi \equiv \partial \tilde{m}/\partial h|_{h=0}$ is infinite, so we need to compute in the ordered phase. Consider $F$ for $h \neq 0$. $F'$ vanishes for $bm^2 = hm^3 - at$, differentiating this with respect to $h$ and then setting $h = 0$ gives $2bm\chi = m^{-3}$ so $\chi \equiv m^{-4} \div t^{-2}$, so $\gamma = 2$.

$\delta$ is defined by $m|_{t=0} \sim h^{1/\delta}$. For $t = 0$, $F'' = -h + bm^5$, which vanishes for $m = (h/b)^{1/5}$, so $\delta = 5$.

These values of $\alpha, \beta, \gamma, \delta$ satisfy the stated relations.

Prob. 4. In lecture we considered ferromagnets, where nearest neighbor spins have lower energy if they are aligned. This problem is about antiferromagnets, where neighboring spins line up in opposite directions, so that the total magnetization is zero. Mean field theory gives a description of antiferromagnetism that is directly analogous to the ferromagnetism that we discussed.

(a) Construct an appropriate mean field theory for the antiferromagnetic Ising model.

(b) Calculate the magnetic susceptibility for this system.

The trick to handling an Ising model for an antiferromagnetic system in the mean field approximation is to regard the lattice as decomposed into two sublattices, say ‘1’ and ‘2’, such that the nearest neighbors of a site on lattice ‘1’ are on ‘2’, and vice versa. Then the usual mean-field argument leads to two coupled equations for the mean spin values

$$\langle s_i \rangle = \sigma_1 , \sigma_2 , \text{ according to whether site } i \text{ is on sublattice ‘1’ or ‘2’}$$
\[ \sigma_1 = \tanh(\eta + \epsilon \sigma_2), \quad \sigma_2 = \tanh(\eta + \epsilon \sigma_1) \quad \text{where} \quad \eta = \beta \mu H, \quad \epsilon = \beta J z. \]

For \( H = 0 \), i.e. \( \eta = 0 \), a sketch of the graphs of both equations on the \((\sigma_1, \sigma_2)\) plane immediately shows:

For \( |\epsilon| < 1 \) (i.e. \( T > T_c \), where \( k_B T_c = |J| z \)) the only solution is \( \sigma_1 = \sigma_2 = 0 \).

For \( |\epsilon| > 1 \) (i.e. \( T < T_c \)) there is an additional pair of solutions – which are the physical solutions because they minimize the free energy – namely for \( \epsilon > 1 \) (i.e. \( J > 0 \), ‘ferromagnetic coupling’):

\[ \sigma_1 = \sigma_2 = \sigma, \quad \text{where} \quad \sigma = \tanh(\epsilon \sigma) \]

for \( \epsilon < -1 \) (i.e. \( J < 0 \), ‘antiferromagnetic coupling’):

\[ \sigma_1 = -\sigma_2 = \sigma, \quad \text{where} \quad \sigma = \tanh(-\epsilon \sigma). \]

The magnetic susceptibility (per site) is \( \beta \mu \chi \), where

\[ \chi = \frac{1}{2}(\alpha' + \beta') \quad \text{where} \quad \alpha' = \frac{\partial}{\partial \eta}|_{\eta=0}. \]

Taking \( \partial / \partial \eta \) of the two coupled equations gives

\[ \sigma'_1 = \frac{(1 + \epsilon \sigma_2')}{\cosh^2(\epsilon \sigma_2)}, \quad \sigma'_2 = \frac{(1 + \epsilon \sigma_1')}{\cosh^2(\epsilon \sigma_1)}; \]

adding these equations together, using \( \sigma_1 = \pm \sigma_2 \) for \( \eta = 0 \), gives

\[ \chi = \frac{(1 + \epsilon \chi)}{\cosh^2(\epsilon \sigma)}, \]

so

\[ \chi = (\cosh^2(\epsilon \sigma) - \epsilon)^{-1} \]

where \( \sigma = 0 \), for \(|\epsilon| < 1 \); \( \sigma = \tanh(|\epsilon| \sigma) \), for \(|\epsilon| > 1 \).

One sees:

For \( T > T_c \) (i.e. \(|\epsilon| < 1 \)), \( \chi = (1 - \epsilon)^{-1} \).

For \( T \to T_c^- \) (i.e. \(|\epsilon| \to 1^+ \)):

for \( J > 0 \) (ferromagnetic), \( \chi \to \frac{1}{2}(\epsilon - 1)^{-1} \)

for \( J > 0 \) (antiferromagnetic), \( \chi \to \frac{1}{2} \).