PROBLEM 1 (Leslie and Robin each has a stockpile of $M$ symbols that can be used to make messages. Leslie has $M/4$ of each of 4 different symbols, while Robin has 5 different symbols, in quantities $M/2$, $M/4$, $M/8$, $M/16$, and $M/16$. You may assume that $M$ is very large.

a) (12 points) Let $\mathcal{N}_L$ and $\mathcal{N}_R$ be the number of distinct messages of length $M$ that can be constructed by Leslie and Robin, respectively (each message uses every symbol in the relevant stockpile). Find

$$\lim_{M\to\infty} \frac{1}{M} \ln \left[ \frac{\mathcal{N}_L}{\mathcal{N}_R} \right].$$

(1)

b) (4 points) How many bits of information are in a message of length $M$ that Leslie composes? (A message of length 1, where the symbol is one of two equally likely possibilities, has one bit of information.)

c) (4 points) How many bits of information are in a message of length $M$ that Robin composes?

Solution:

a) The numbers of messages that Leslie and Robin can construct are

$$\mathcal{N}_L = \frac{M!}{[(M/4)!]^4}, \quad \mathcal{N}_R = \frac{M!}{(M/2)(M/4)(M/8)(M/16)(M/16)!}.$$ 

Using Stirling’s formula, one finds

$$\lim_{M\to\infty} \ln \mathcal{N}_L = M \ln M - M - 4(M/4)\ln(M/4) + 4(M/4) = 2M$$

and

$$\lim_{M\to\infty} \ln \mathcal{N}_R = M \ln M - M - (M/2)\ln(M/2) + M/2 - (M/4)\ln(M/4) + M/4$$

$$- (M/8)\ln(M/8) + M/8 - 2(M/16)\ln(M/16) + 2(1/16)$$

$$= 15M/8,$$

so

$$\lim_{M\to\infty} \frac{1}{M} \ln \frac{\mathcal{N}_L}{\mathcal{N}_R} = \frac{1}{8} \ln 2.$$
b) This part can be done several ways. One way is to note that the actual message sent is one of \( \mathcal{N}_L \) equally likely possibilities, so the total information in the message is \( \log_2 \mathcal{N}_L = 2M \). (The units are set by noting that one gets one bit if there are two equally likely possibilities.) In other words, the information is the log of the total number of messages divided by log of 2. The other method is to calculate the information per symbol as \( -\sum_{i=1}^{1} p_i \log_2 p_i = -\sum_{i=1}^{1} (1/4) \log_2 (1/4) = 2 \), so that the information in a message of length \( M \) is \( 2M \).

c) The method is identical to part b). We can either calculate \( \ln \mathcal{N}_R / \ln 2 \) or find the information per symbol using Shannon’s formula, which says that the information per symbol is

\[
- \sum_{i=1}^{5} p_i \ln_2 p_i = - \frac{1}{2} \ln_2 (1/2) - \frac{1}{4} \ln_2 (1/4) - \frac{1}{8} \ln_2 (1/8) - \frac{2}{16} \ln_2 (1/16) = \frac{15}{8}.
\]

Either way, we find that the information in a message with \( M \) symbols sent by Robin is \( 15M/8 \) bits.

**PROBLEM 2** (Show that if one assumes the functional form

\[
S = \sum_{\nu} P_{\nu} f(P_{\nu}),
\]

where \( f(x) \) is some function of \( x \), then the requirement that \( S \) is extensive implies that \( f(x) = c \ln x \), where \( c \) is an arbitrary constant.)

Extensiveness implies:

\[
S[\{P_{\mu,}\}] = S[\{P_{\mu}\}] + S[\{P_{\nu}\}]
\]

for independent ensembles. This also implies that the probability distribution of the two systems is independent: \( P_{\nu,\mu} = P_{\nu} P_{\mu} \) for all \( \mu, \nu \). Applying this condition on the function given, one has:

\[
S[\{P_{\nu,}\}] = \sum_{\mu,\nu} P_{\mu} P_{\nu} f(P_{\mu} P_{\nu}) = S[\{P_{\mu}\}] + S[\{P_{\nu}\}] = \sum_{\mu} P_{\mu} f(P_{\mu}) + \sum_{\nu} P_{\nu} f(P_{\nu}).
\]

Using the fact that the probability distributions are normalized, one finds that

\[
f(P_{\mu} P_{\nu}) = f(P_{\mu}) + f(P_{\nu}),
\]

which defines, under the assumption of differentiability, the logarithm function up to a multiplicative constant (as derived in lecture):

\[
f(xy) = f(x) + f(y)
\]

\[
\partial_y : \quad x f'(xy) = f'(y)
\]

\[
y = 1 : \quad x f' = c \quad [= f'(1)]
\]

\[
\int dx : \quad f = c \ln x + d
\]

substitute in original eqn : \( d = 0 \).
PROBLEM 3 (Consider a system with two states, 1 and 2, where the probability of being in state 1 is given by

\[ P_1 = \frac{h_1}{h_1 + h_2} \]

and the probability of being in state 2 is given by

\[ P_2 = \frac{h_2}{h_1 + h_2} \]

and the positive numbers \( h_1 \) and \( h_2 \) are assumed known. Determine the fluctuation \( \langle (A - \langle A \rangle)^2 \rangle \) for the stochastic variable \( A \) which takes on the values \( A_1 = +1 \) and \( A_2 = -1 \) in the two states.

Discuss the limit \( h_1 \gg h_2 \).

We have

\[ \langle A \rangle = A(1)P_1 + A(2)P_2 = \frac{h_1 - h_2}{h_1 + h_2}, \text{ and } \langle A^2 \rangle = A(1)^2P_1 + A(2)^2P_2 = \frac{h_1 + h_2}{h_1 + h_2} = 1. \quad (11) \]

So

\[ (\delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 = \frac{4h_1h_2}{(h_1 + h_2)^2}. \quad (12) \]

In the limit \( h_1 \gg h_2 \), one sees that \( \delta A \to 0 \), meaning that, given the system has a much greater probability to be found in the state 1, the observable \( A \) doesn’t fluctuate much.

PROBLEM 4 (Consider an \( n \) state system where the states, labelled \( \ell = 1, 2, 3, \ldots, n \), are equally probable. In this system there are two stochastic variables of interest:

\[ A_\ell = A_0 \cos \left( \frac{2\pi \ell}{n} \right) \]

and

\[ B_\ell = B_0 \ell/n \]

where \( A_0 \) and \( B_0 \) are constants. Find the correlation function \( C_{AB} \), where

\[ C_{AB} = \langle \delta A \delta B \rangle \equiv \langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle \],

as a function of \( n \).

Using the formulas:

\[ \sum_{l=1}^{n} \cos \left( \frac{2\pi \ell}{n} \right) = \text{Re} \left( \sum_{l=1}^{n} e^{\frac{2\pi i \ell}{n}} \right) = \text{Re} \frac{e^{\frac{2\pi i}{n}} - e^{\frac{2\pi i (n+1)}{n}}}{1 - e^{\frac{2\pi i}{n}}} = 0, \quad (14) \]
and

$$\sum_{l=1}^{n} l \cos \left( \frac{2\pi l}{n} \right) = \frac{\partial}{\partial x} \left( \text{Im} \sum_{l=1}^{n} e^{i2\pi l} \right) \bigg|_{x=\frac{2\pi}{n}} = \frac{n}{2},$$

valid for \( n > 1 \), which can be seen by direct computation. One then sees immediately that:

$$C_{AB} = \langle AB \rangle - \langle A \rangle \langle B \rangle = \frac{A_0 B_0}{n^2} \sum_{l=1}^{n} l \cos \left( \frac{2\pi l}{n} \right) = \frac{A_0 B_0}{2n} \quad (n \geq 1) \quad (16)$$

$$= 0 \quad (n = 0). \quad (17)$$