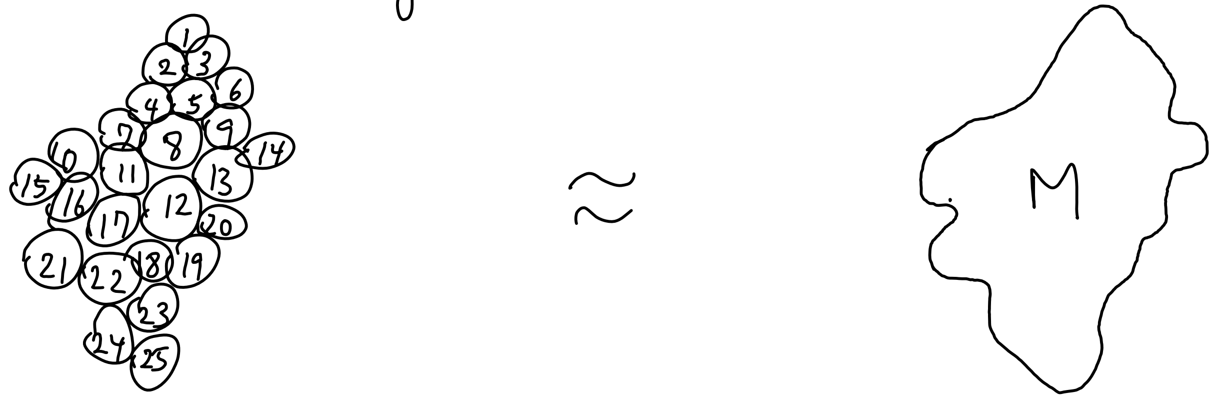
Chapter 8: System of Particles and Conservation of Linear Momentum

Main idea:
Motion of system of particles = center of mass motion + motion of various parts of the system relative to the center of mass

System of particles:

e.g. extended objects

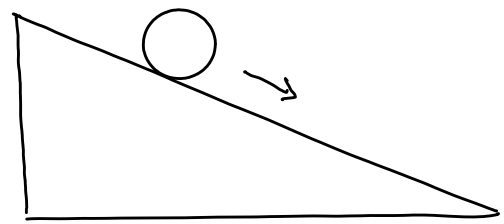


$$M = \sum_{i=1}^{25} m_i$$

e.g. Collection of "particles"

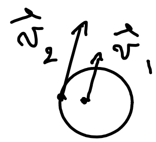
Bulk motion is often of interest.

e.g.



"Bulk" motion here means the coordinate of the center of the ball as a function of time.

contrast: "internal" motion



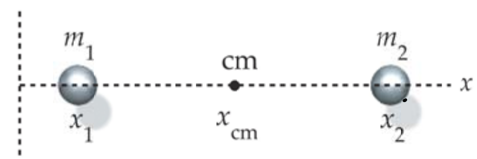
$\vec{v}_1$ and $\vec{v}_2$ represent velocities of two different parts of a ball arising from "spinning" motion about the center.

A useful choice for bulk motion description: center of mass
Allows following statement (as we will see): The total external force equals the total mass times the acceleration of the center of mass.

Center of mass

$$\vec{r}_{cm} \equiv \frac{\sum_i m_i \vec{r}_i}{M} \quad \text{where} \quad M \equiv \sum_i m_i$$

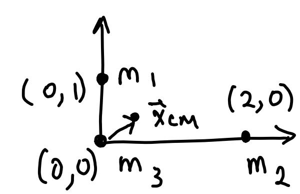
Example

Find the CM of 

Ans

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Example

Find the CM of 

Ans

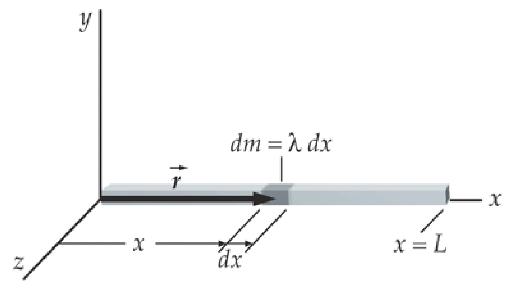
$$\vec{x}_{cm} = \frac{(0,1)m_1 + (2,0)m_2 + (0,0)m_3}{m_1 + m_2 + m_3}$$

$$= \left(\frac{2m_2}{m_1+m_2+m_3} , \frac{m_1}{m_1+m_2+m_3} \right)$$

Suppose the mass distribution is continuous

$$\vec{r}_{CM} = \frac{1}{M}\int dm\, \vec{r} \qquad \text{where} \qquad M \equiv \int dm$$

Example



$$\vec{r} = x\,\hat{x}$$

$dm = \lambda\, dx$
where
$\lambda \equiv$ mass per unit length.

$$\therefore \quad M = \int dm = \lambda \int_0^L dx = \lambda L$$

$$\vec{r}_{CM} = \frac{1}{\lambda L}\int_0^L (dx\,\lambda)(x\,\hat{x})$$

$$= \frac{\hat{x}}{L}\int_0^L dx\, x = \frac{\hat{x}}{L}\,\frac{1}{2}x^2\Big|_0^L$$

$$= \boxed{\frac{\hat{x}}{2}\,L}$$

intuitively correct.

Motion of CM

$$\frac{d^2\vec{r}_{CM}}{dt^2} = \frac{\sum_i m_i \frac{d^2\vec{r}_i}{dt^2}}{M} = \frac{\sum_i \overbrace{\left[\left(\sum_j \vec{F}_{ij}\right) + \vec{F}_{ext\,i}\right]}^{\text{Newton's second law on "point" particles}}}{M}$$

$\vec{F}_{ij}$: jth particle exerting force on ith particle.

Newton's 3rd law: $\vec{F}_{ij} = -\vec{F}_{ji}$

$$\therefore \sum_i \sum_j \vec{F}_{ij} = \frac{1}{2}\left\{\sum_i \sum_j \vec{F}_{ij} + \sum_j \sum_i \vec{F}_{ji}\right\}$$

$$= \frac{1}{2}\left\{\sum_{ij}\left(\vec{F}_{ij} - \vec{F}_{ij}\right)\right\} = 0$$

$$\therefore \frac{d^2\vec{r}_{CM}}{dt^2} = \frac{1}{M}\sum_i \vec{F}_{ext\,i} \Rightarrow \boxed{\vec{F}_{ext\,total} = M\vec{a}_{CM}}$$

$\boxed{\text{As advertised!}}$

We have been implicitly been using this when we applied Newton's Laws to semi-realistic systems.

Linear momentum

$$\vec{p} \equiv m\vec{v} \qquad \text{for pointlike particles.}$$

Note that Newton's second law can be concisely written as

$$\boxed{\vec{F}_{tot} = \frac{d\vec{P}}{dt}}$$

Conservation of linear momentum

Consider the total momentum (sum over constituents)

$$\vec{P}_{tot} = \sum_i \vec{P}_i$$

$$\frac{d}{dt}\vec{P}_{tot} = \frac{d}{dt}\sum_i m_i \vec{v}_i = \sum_i m_i \frac{d}{dt}\vec{v}_i = \sum_i m_i \frac{d^2}{dt^2}\vec{r}_i$$

$$= \frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \frac{d^2}{dt^2}\left(M\vec{r}_{CM}\right) = M\frac{d^2\vec{r}_{CM}}{dt^2}$$

$$= \vec{F}_{ext_{total}}$$

Suppose one is interested in a closed system of particles.

$\Rightarrow$ no external forces $\Longrightarrow$ $\frac{d}{dt}\vec{P}_{tot} = 0$ $\Rightarrow$ $\boxed{\vec{P}_{tot} = \text{const}}$

Conservation!

Example

Suppose an unstable nucleus at rest decays into two particles of masses m_1 and m_2. Suppose one measures the velocity of m_2 to be $\vec{v}_{f_2}$. What is the velocity of m_1?

Answer: Momentum conservation

$$\underbrace{0}_{\text{before}} = \underbrace{m_1 \vec{v}_{f_1} + m_2 \vec{v}_{f_2}}_{\text{after}}$$

momentum before since nucleus is not moving

final velocity 1

final velocity 2.

Kinetic energy

Consider the Total kinetic energy of a system of particles :

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)$$

Let $\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt}$ Let $\vec{u}_i \equiv \vec{v}_i - \vec{v}_{CM}$

$$K = \sum_i \frac{1}{2} m_i (\vec{v}_{CM} + \vec{u}_i)\cdot(\vec{v}_{CM} + \vec{u}_i) = \sum_i \frac{1}{2} m_i (\vec{v}_{CM}\cdot\vec{v}_{CM} + 2\vec{v}_{CM}\cdot\vec{u}_i + \vec{u}_i\cdot\vec{u}_i)$$

$$= \frac{1}{2}\left(\sum_i m_i\right)|\vec{v}_{CM}|^2 + \vec{v}_{CM}\cdot\left(\sum_i m_i \vec{u}_i\right) + \sum_i \frac{1}{2} m_i \vec{u}_i\cdot\vec{u}_i$$

$$\sum_i m_i \vec{u}_i = \sum_i m_i (\vec{v}_i - \vec{v}_{CM}) = \sum_i m_i \left(\vec{v}_i - \frac{d}{dt}\vec{r}_{CM}\right)$$

$$= \sum_i m_i \left(\vec{v}_i - \frac{d}{dt}\frac{1}{M}\sum_j m_j \vec{r}_j\right) = \sum_i m_i \left(\vec{v}_i - \frac{1}{M}\sum_j m_j \vec{v}_j\right)$$

$$= \sum_i \left(\frac{\vec{v}_i m_i \sum_j m_j - \sum_j m_i m_j \vec{v}_j}{M}\right) = \frac{1}{M}\left(\sum_{ij}(m_i m_j \vec{v}_i - m_i m_j \vec{v}_j)\right) = 0$$

$$\therefore \boxed{K = \frac{1}{2} M |\vec{v}_{CM}|^2 + \underbrace{\sum_i \frac{1}{2} m_i |\vec{u}_i|^2}_{\text{relative kinetic energy.}}}$$

In the absence of external forces $\frac{d\vec{v}_{CM}}{dt} = 0 \Rightarrow$ only relative kinetic energy can change.

Collision

two or more objects interact to change the momenta of one another (interactions usually last a "short" duration).
Typically, concerned with "before" and "after".

elastic collision = kinetic energy conserved.
inelastic " = " " not conserved

Momentum is always conserved in the absence of external forces.
Kinetic energy may or may not be conserved even in the absence of external forces

example

Suppose $\overset{1}{\underset{m}{\bigcirc}} \rightarrow v_i$ $\overset{2}{\underset{m}{\bigcirc}}$ at rest (restrict the motion to 1D)

What are the speeds of 1 and 2 after the elastic collision?