Physics 247
Lecture 5

September 15, 2006

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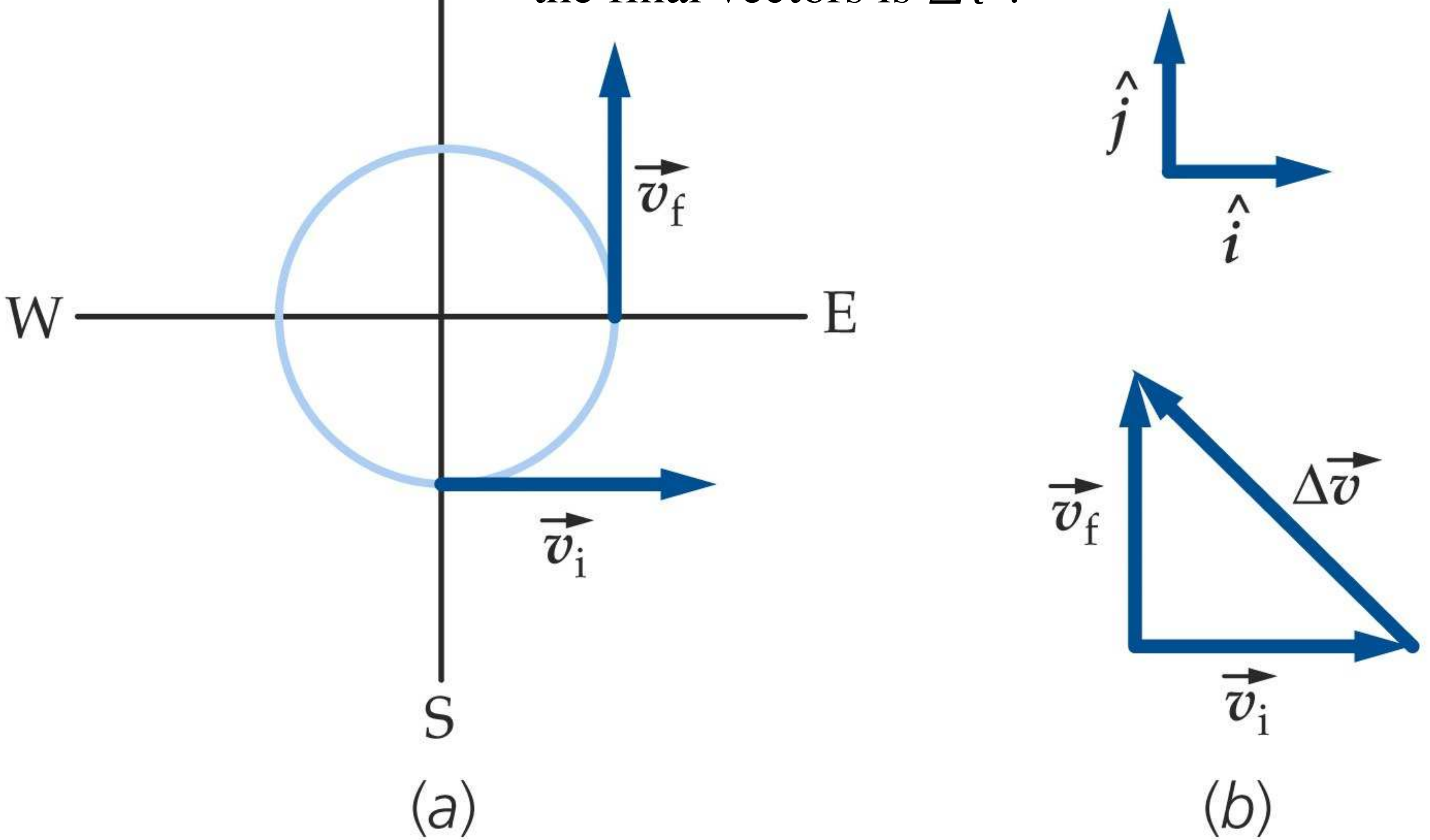
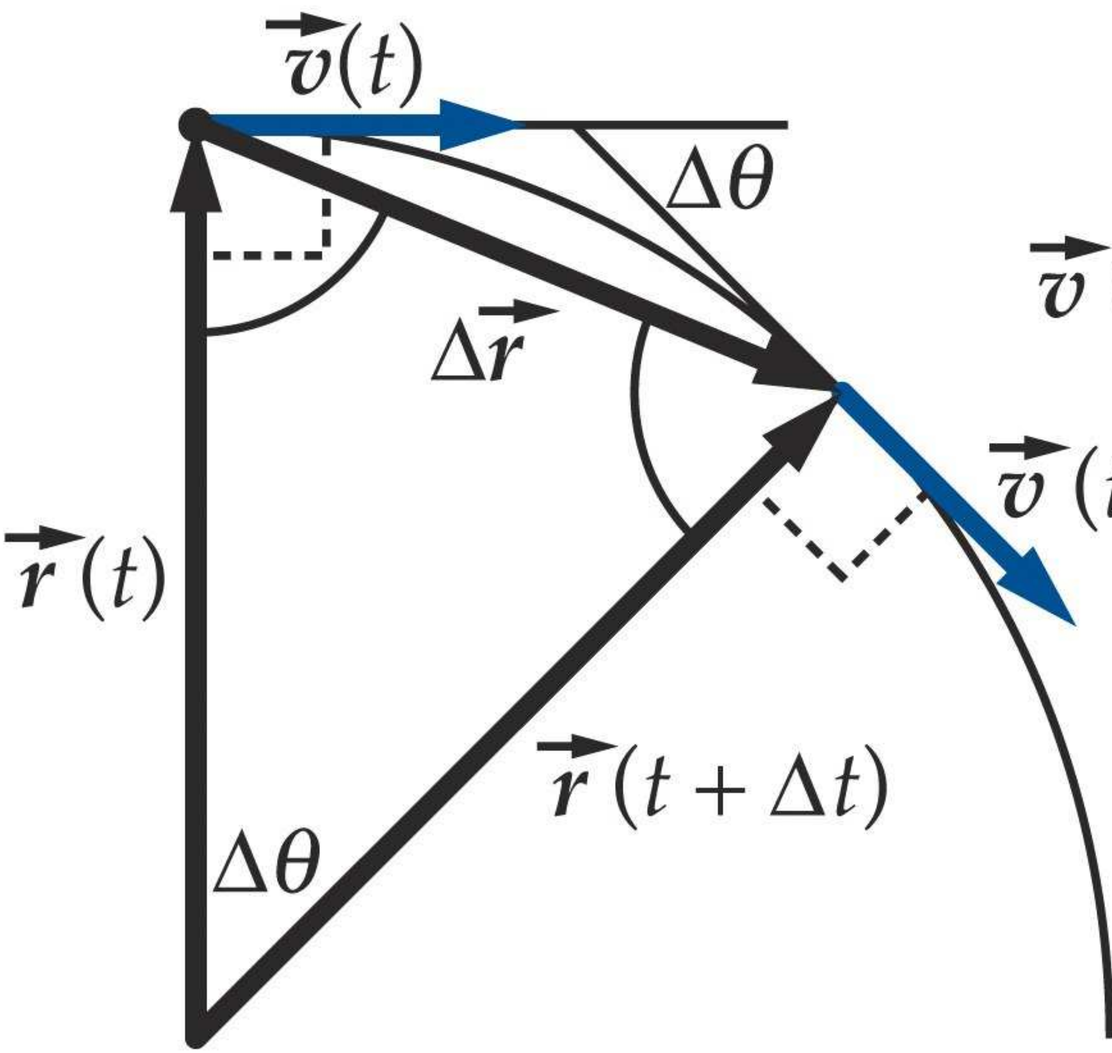
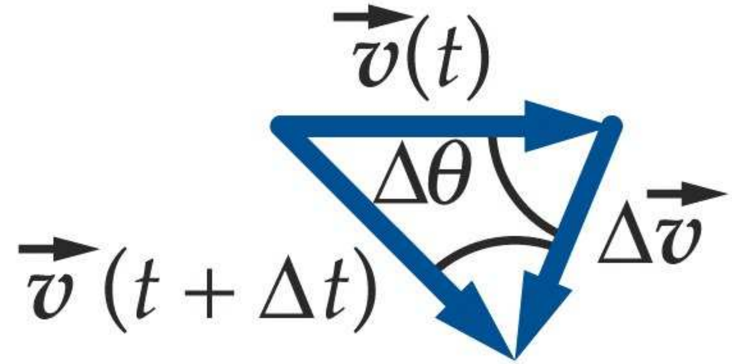
Circular Motion

What is the average acceleration if the time interval between the initial and the final vectors is Δt ?



Uniform circular motion: derivation of $|\vec{a}| = \dfrac{v^2}{r}$



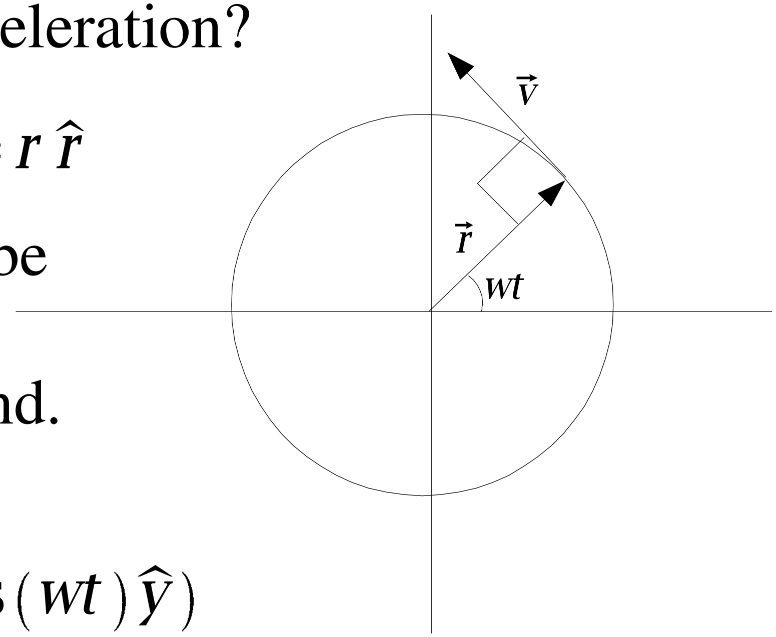


$$\frac{|\Delta\vec{v}|}{|\vec{v}|} = \frac{|\Delta\vec{r}|}{|\vec{r}|} = \frac{|\vec{v}_{av}|\Delta t}{|\vec{r}|}$$

$$\frac{|\Delta\vec{v}|}{\Delta t} = \frac{|\vec{v}_{av}||\vec{v}|}{|\vec{r}|}$$

$$|\vec{a}| = \lim_{\Delta t \to 0} \frac{|\Delta\vec{v}|}{\Delta t} = \frac{|\vec{v}|^2}{|\vec{r}|}$$

What about the direction of acceleration?



const

$$\vec{r} = r\left(\cos(wt)\,\hat{x} + \sin(wt)\,\hat{y}\right) = r\,\hat{r}$$

$|\vec{r}| = r$ constant as it should be

w has units of radians per second.

$$\vec{v} = \frac{d}{dt}\vec{r} = r\,w\left(-\sin(wt)\,\hat{x} + \cos(wt)\,\hat{y}\right)$$

$|\vec{v}| = r\,w$ constant as it should be

$$\boxed{\vec{a} = \frac{d^2}{dt^2}\vec{r} = -r\,w^2\left(\cos(wt)\,\hat{x} + \sin(wt)\,\hat{y}\right) = -w^2\,\vec{r}}$$

Hence, this is called the centripetal acceleration!

$|\vec{a}| = r\,w^2 = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$ as we saw in the last slide.

Bonus: We know the direction (see yellow box)!

Exercise

We have seen in the last slide that

$$\vec{r} = r\left(\cos(wt)\,\hat{x} + \sin(wt)\,\hat{y}\right)$$

$$\vec{v} = r\,w\left(-\sin(wt)\,\hat{x} + \cos(wt)\,\hat{y}\right)$$

a) Show that $\vec{r}$ and $\vec{v}$ are perpendicular.

b) Show that if $\hat{\theta}$ is defined to be the unit vector perpendicular to $\hat{r}$ in the direction of increasing polar angle on the 2D plane, find A and B in the expression

$$\hat{\theta} = A\,\hat{x} + B\,\hat{y}$$

c) In uniform circular motion, write $\dfrac{d\hat{r}}{dt}$ in terms of $\hat{r}$ and $\hat{\theta}$.

More Exercises

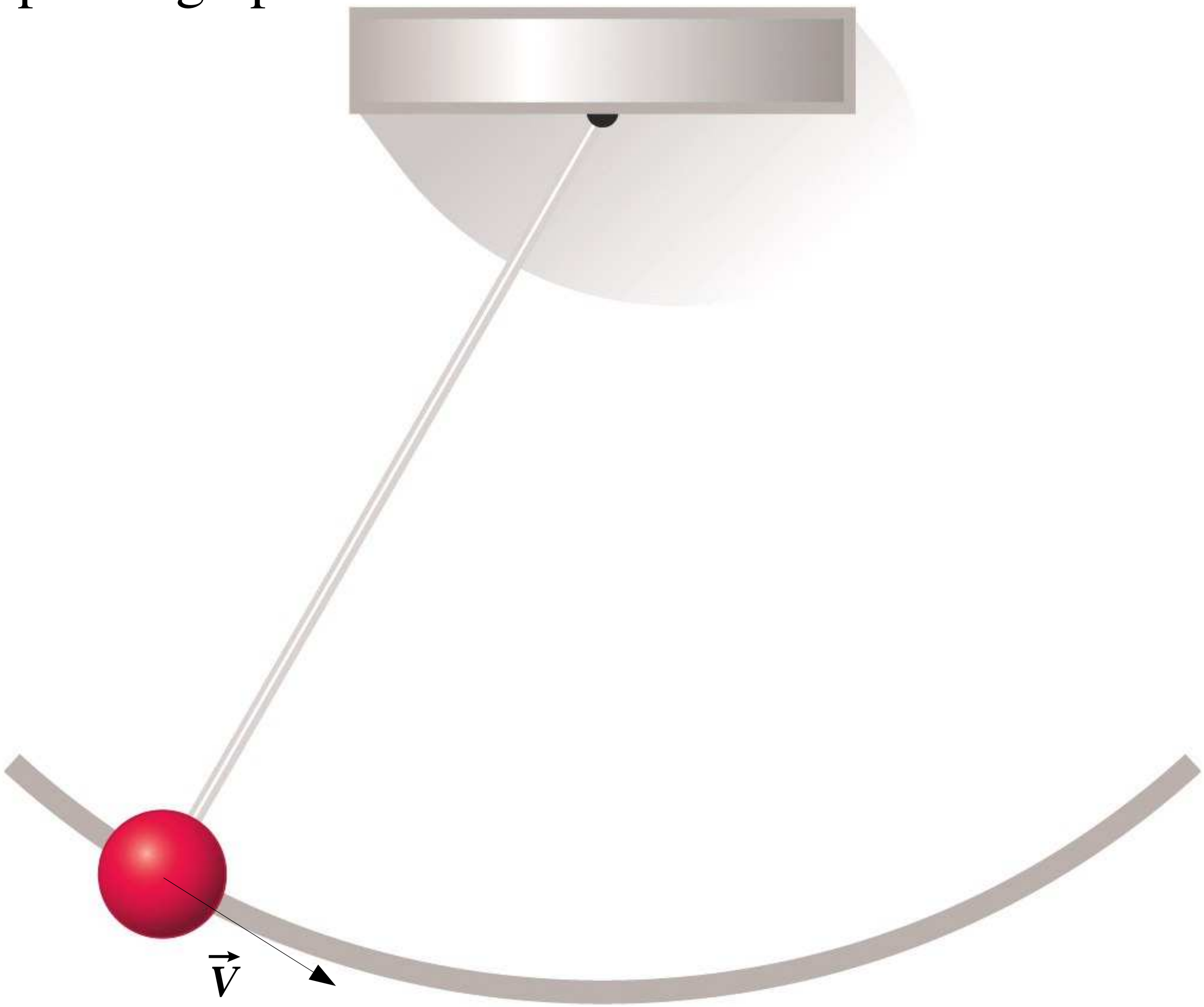
You have learned that for uniform circular motion that

$$\frac{d\,\hat{r}}{dt} = w\,\hat{\theta}$$

From the fact that $\dfrac{d^2\vec{r}}{dt^2} = -w^2\vec{r}$ for uniform circular motion, deduce

$$\frac{d\,\hat{\theta}}{dt}$$

Which direction is the (instantaneous) acceleration if you know it is speeding up here?

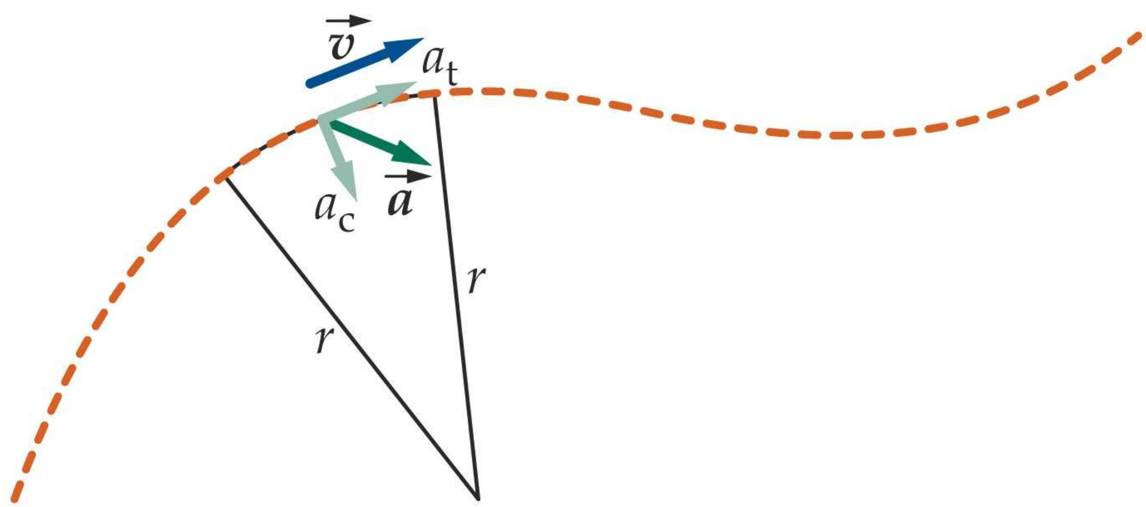


$$a = a_c = \frac{v^2}{r} \qquad \text{3-24}$$

CENTRIPETAL ACCELERATION

$$a_t = \frac{dv}{dt} \qquad \text{3-26}$$

TANGENTIAL ACCELERATION



Proof

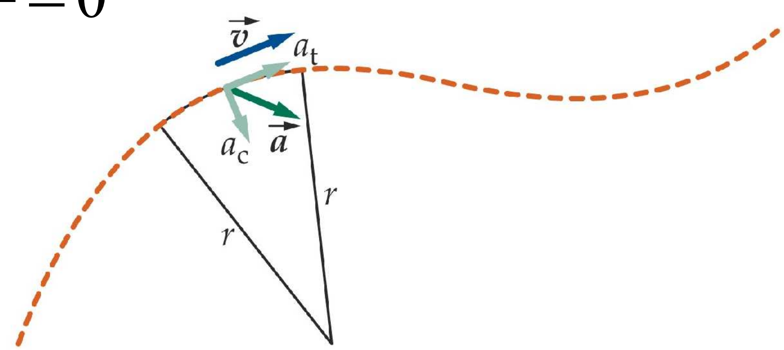
$$\vec{r} = r\,\hat{r} \qquad \hat{r} = \cos[\theta(t)]\,\hat{x} + \sin[\theta(t)]\,\hat{y}$$

$$\vec{v} = \frac{d}{dt}\vec{r} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \qquad \frac{d\hat{r}}{dt} = \left(\frac{d\theta}{dt}\right)\left(-\sin[\theta]\,\hat{x} + \cos[\theta]\,\hat{y}\right) = \left(\frac{d\theta}{dt}\right)\hat{\theta}$$

$$\vec{a} = \frac{d^2}{dt^2}\vec{r} = \frac{d^2 r}{dt^2}\hat{r} + 2\frac{dr}{dt}\frac{d\hat{r}}{dt} + r\frac{d^2\hat{r}}{dt^2}$$

Use this to evaluate $\frac{d\hat{\theta}}{dt}$

If we match the arc with arc of a circle with radius r, we have at that location $\frac{dr}{dt} = 0 \qquad \frac{d^2 r}{dt^2} = 0$



$$\vec{a} = r\frac{d^2\hat{r}}{dt^2} = \underbrace{r\frac{d^2\theta}{dt^2}}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt}$$

$$= a_t\hat{\theta} - \vec{r}\left(\frac{d\theta}{dt}\right)^2 = a_t\hat{\theta} - \hat{r}\,a_c$$

Example

Calculate $\vec{a}_t$ and $\vec{a}_c$ in terms of the variables shown: