Taylor series

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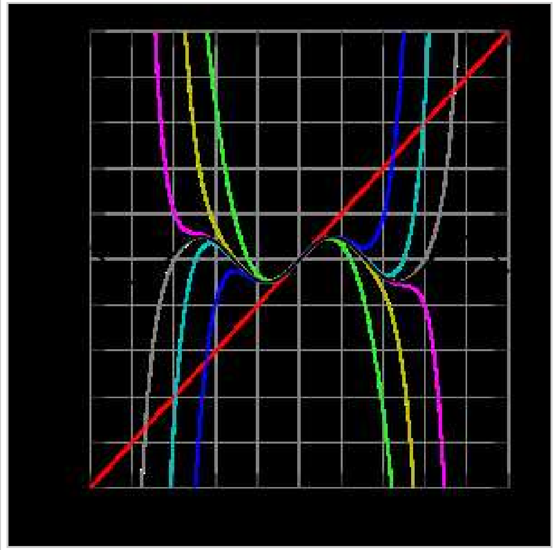
In mathematics, the **Taylor series** of an infinitely differentiable real (or complex) function *f*, defined on an open interval $(a - r, a + r)$, is the power series shown below. The series is named in honor of English mathematician Brook Taylor.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Here, $n!$ is the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f at the point a. If $a = 0$, the series is also called a **Maclaurin series**, named after Scottish mathematician Colin Maclaurin.

Functions that involve rational operations such as addition, subtraction, multiplication and division are relatively easy to evaluate. Many other functions aren't so easy to evaluate, like those that involve logarithms or trigonometric functions such as $\cos(x)$. These and many other functions are approximately equal to their **Taylor series** within a certain range and so the partial sums of this series can be used as a good approximation.



As the degree of the Taylor series rises, it approaches the correct function. This image shows sin*x* and Taylor approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13.

Pictured in the right are increasingly accurate approximations of $\sin(x)$ around the point $a = 0$. The yellow curve is a polynomial of degree seven:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Contents

History

The Taylor series, power series, and infinite series expansions of functions were first discovered in India by Madhava in the 14th century. He found a number of special cases of the Taylor series, including the Taylor series for the trigonometric functions of sine, cosine, tangent and arctangent, and the second-order Taylor series approximations of the sine and cosine functions, which he extended to the third-order Taylor series approximation of the sine function. He also discovered the power series of the radius, diameter, circumference, angle θ, π and π/4, along with rational approximations of π, and infinite continued fractions. His students and followers in the Kerala School further expanded his works with various series expansions and rational approximations until the 16th century.

In the 17th century, James Gregory also worked in this area and published several Maclaurin series. It was not until 1715 however that a general method for constructing these series for all functions for which they exist was finally provided by Brook Taylor, after whom the series are now named.

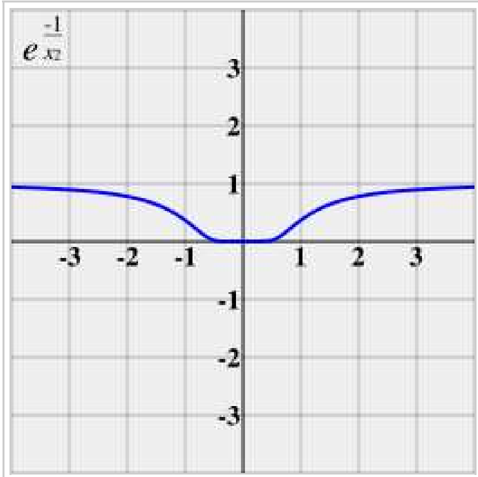
The Maclaurin series was named after Colin Maclaurin, a professor in Edinburgh, who published the special case of the Taylor result in the 17th century.

Properties

If this series converges for every x in the interval $(a - r, a + r)$ and the sum is equal to $f(x)$, then the function $f(x)$ is said to be **analytic in the interval** $(a - r, a + r)$. If this is true for any r then the function is said to be **analytic**. To check whether the series converges towards $f(x)$, one normally uses estimates for the remainder term of Taylor's theorem. A function is analytic if and only if it can be represented as a power series; the coefficients in that power series are then necessarily the ones given in the above Taylor series formula.

The importance of such a power series representation is at least fourfold. First, differentiation and integration of power series can be performed term by term and is hence particularly easy. Second, an analytic function can be uniquely extended to a holomorphic function defined on an open disk in the complex plane, which makes the whole machinery of complex analysis available. Third, the (truncated) series can be used to compute function values approximately (often by recasting the polynomial into the Chebyshev form and evaluating it with the Clenshaw algorithm). Fourth, algebraic operations can often be done much more readily on the power series representation; for instance the simplest proof of Euler's formula uses the Taylor series expansions for sine, cosine, and exponential functions. This result is of fundamental importance in such fields as harmonic analysis.

Note that there are examples of infinitely differentiable functions $f(x)$ whose Taylor series converge, but are *not* equal to $f(x)$. For instance, for the function defined piecewise by saying that $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$, all the derivatives are zero at $x = 0$, so the Taylor series of $f(x)$ is zero, and its radius of convergence is infinite, even though the function most definitely is not zero. This particular pathology does not afflict complex-valued functions of a complex variable. Notice that e^{-1/z^2} does not approach 0 as z approaches 0 along the imaginary axis.



The function e^{-1/x^2} is not analytic: the Taylor series is identically 0, although the function is not.

Some functions cannot be written as Taylor series because they have a singularity; in these cases, one can often still achieve a series expansion if one allows also negative powers of the variable x; see Laurent series. For example, $f(x) = e^{-1/x^2}$ can be written as a Laurent series.

The Parker-Sochacki method is a recent advance in finding Taylor series which are solutions to differential equations. This algorithm is an extension of the Picard iteration.

Taylor series for several variables

The Taylor series may also be generalised to functions of more than one variable with

$$T(x_1,\cdots,x_d) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_d=0}^{\infty}\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_d}}{\partial x_d^{n_d}}\frac{f(a_1,\cdots,a_d)}{n_1!\cdots n_d!}(x_1-a_1)^{n_1}\cdots(x_d-a_d)^{n_d}$$

For example, for a function that depends on two variables, *x* and *y*, the Taylor series to second order about the point (*a*, *b*) is:

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right). \end{aligned}$$

A second-order Taylor series expansion of a scalar-valued function of more than one variable can be compactly written as

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a})^T(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T\nabla^2 f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \cdots$$

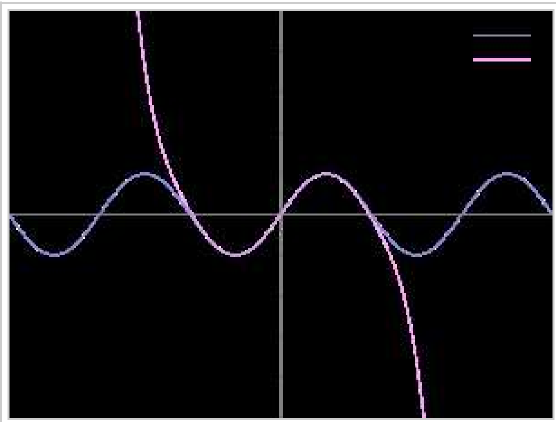
where $\nabla f(\mathbf{a})$ is the gradient and $\nabla^2 f(\mathbf{a})$ is the Hessian matrix (not to be confused with the Laplacian, which sometimes has the same notation). Applying the multi-index notation the Taylor series for several variables becomes

$$T(\mathbf{x}) = \sum_{|\alpha|\ge 0}\frac{\mathrm{D}^\alpha f(\mathbf{a})}{\alpha!}(\mathbf{x}-\mathbf{a})^\alpha$$

in full analogy to the single variable case.

List of Taylor series of some common functions

Several important Taylor/Maclaurin series expansions follow. All these expansions are also valid for complex arguments *x*.


The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full cycle centered on the origin.

Exponential function and natural logarithm:

$$\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\quad\text{for all } x$$

$$\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}\quad\text{for } |x|<1$$

Geometric series:

$$\frac{x^m}{1-x} = \sum_{n=m}^{\infty}x^n\quad\text{for } |x|<1$$

Binomial theorem:

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n\quad\text{for all } |x|<1\quad\text{and all complex } \alpha$$

Trigonometric functions:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \quad = x + \frac{x^3}{3} + \frac{2x^5}{15} + .. \text{ for } |x| < \frac{\pi}{2}$$

where the *B*s are Bernoulli numbers.

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| < 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| < 1$$

Hyperbolic functions:

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \quad \text{for all } x$$

$$\tanh(x) = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} \quad \text{for } |x| < \frac{\pi}{2}$$

$$\operatorname{arcsinh}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| < 1$$

$$\operatorname{arctanh}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} \quad \text{for } |x| < 1$$

Lambert's W function:

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \quad \text{for } |x| < \frac{1}{\mathrm{e}}$$

The numbers B_k appearing in the *summation* expansions of tan(*x*) and tanh(*x*) are the Bernoulli numbers. The binomial expansion uses binomial coefficients. The E_k in the expansion of sec(*x*) are Euler numbers.

Calculation of Taylor series

Several methods exist for the calculation of Taylor series of a large number of functions. One can attempt to use the Taylor series as-is and generalize the form of the coefficients, or one can use manipulations such as substitution, multiplication or division, addition or subtraction of standard Taylor series to construct the Taylor series of a function, by virtue of Taylor series being power series. In some cases, one can also derive the Taylor series by repeatedly applying integration by parts.

For example, consider the function

$$f(x) = \ln(1+\cos x)$$

for which we want a Taylor series about 0.

We have:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x| < 1$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

We can simply substitute the second series into the first. Doing so,

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^2 + \frac{1}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^3 - \cdots$$

Expanding by using multinomial coefficients gives the requisite Taylor series.

Or, for example, consider

$$g(x) = \frac{e^x}{\cos x}$$

We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Then,

$$\frac{e^x}{\cos x} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Assume the power series is

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Then

$$\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$= c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{4!}x^4 + c_1 x - \frac{c_1}{2}x^3 + \frac{c_1}{4!}x^5 + c_2 x^2 - \frac{c_2}{2}x^4 + \frac{c_2}{4!}x^6 + c_3 x^3 - \frac{c_3}{2}x^5 + \frac{c_3}{4!}x^7 + \cdots$$

Collecting the terms up to fourth order yields

$$= c_0 + c_1 x + \left(c_2 - \frac{c_0}{2}\right)x^2 + \left(c_3 - \frac{c_1}{2}\right)x^3 + \left(c_4 + \frac{c_0}{4!} - \frac{c_2}{2}\right)x^4 + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Comparing coefficients finally yields the Taylor series for the function

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots$$

Taylor series as definitions

Classically the above functions are defined by some property that hold for them, for example the exp function is defined as the function that is equal to its own derivative. However in computable analysis functions must be defined by algorithms rather than properties, so the above Taylor expansions are used as primary definitions rather than derived results. This is also likely to be the case in software implementations of the functions.

See also

- Laurent series
- Taylor's theorem
- Holomorphic functions are analytic — a proof that a holomorphic function can be expressed as a Taylor power series
- Newton's divided difference interpolation
- Madhava of Sangamagrama (credited with the first use of "Taylor" series)

References

- Thomas, George B. Jr.; Finney, Ross L. (1996). *Calculus and Analytic Geometry (9th ed.)*. Addison Wesley. ISBN 0-201-53174-7.
- Greenber, Michael (1998). *Advanced Engineering Mathematics (2nd ed.)*. Prentice Hall. ISBN 0-13-321431-1.

External links

- "Taylor Series" (http://mathworld.wolfram.com/TaylorSeries.html) on MathWorld
- Madhava of Sangamagramma (http://www-groups.dcs.st-and.ac.uk/~history/Projects/Pearce/Chapters/Ch9_3.html)
- "Discussion of the Parker-Sochacki Method" (http://csma31.csm.jmu.edu/physics/rudmin/ParkerSochacki.htm)
- Why so much fuss about Taylor Series Expansion? (http://www.risklatte.com/features/quantsKnow051020.php)

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