PHYSICS 717 PROBLEM SET 10

due: Monday, April 13, 2009, at the beginning of lecture

Problems

1.: Big bang singularity:

a): Show that a diffeomorphism invariant scalar made of curvature tensor diverges for the FRW cosmology with a equation of state $w = 1/3$ for a perfect fluid.

b): Show that the singularity is spacelike.

answer:

(a):

Consider the trace of Einstein tensor:

$$G^a{}_a = R - 2R = -R = 8\pi T$$

where $T \equiv T^a{}_a$. Hence, we can rewrite Einstein's equations as

$$\begin{aligned} R_{ab} &= 8\pi T_{ab} + \frac{1}{2} g_{ab} R \\ &= 8\pi T_{ab} - \frac{1}{2} g_{ab} 8\pi T \\ &= 8\pi (T_{ab} - \frac{1}{2} g_{ab} T). \end{aligned}$$

For a perfect fluid, the stress energy tensor is

$$T_{ab} = (\rho + P) U_a U_b + P g_{ab}$$

while its trace is

$$T = -(\rho + P) + 4P = 3P - \rho.$$

As we learned in class, since for equation of state of $P/\rho = 1/3$, we have $T = 0$, we can neglect the T term in R_{ab} and write

$$\begin{aligned} R_{ab} R^{ab} &= 64\pi^2 (T_{ab} T^{ab}) \\ &= 64\pi^2 [(\rho + P) U_a U_b + P g_{ab}][(\rho + P) U^a U^b + P g^{ab}] \\ &= 64\pi^2 [(\rho + P)(\rho + P) U^a U^b U_a U_b + P g_{ab} (\rho + P) U^a U^b + \\ &\quad + (\rho + P) U_a U_b P g^{ab} + P g_{ab} P g^{ab}] \\ &= 64\pi^2 [(\rho + P)^2 - 2P(\rho + P) + 4P^2] \\ &= 64\pi^2 [\rho^2 + 3P^2] \\ &= 64\pi^2 \rho^2 [\frac{4}{3}] = \frac{256\pi^2}{3} \rho^2. \end{aligned}$$

We derived in class that

$$\rho = \rho_0 (\frac{a_0}{a})^4.$$

Hence, for $a \to 0$, $\rho \to \infty$ and $R_{ab} R^{ab} \to \infty$.

b):

The surface of singularity is spacelike if the normal vector to the surface is timelike. Within the parameterization

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2,$$

the singularity is at $a(t) = 0$, i.e. a constant t hypersurface. Since $g_{0i} = 0$, $(\partial_t)^\mu$ is normal to the constant t hypersurface, and since $(\partial_t)^\mu g_{\mu\nu} (\partial_t)^\nu = -1$, this normal vector is timelike. Hence, the surface of singularity is spacelike.

2.: Show that the spatial part of the FRW metric for $K = 1$ corresponds to the metric of a 3-sphere embedded in Euclidean 4-space. (i.e. Equation for a 3-sphere in Euclidean 4-space is

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

where R is the radius of the 3-sphere.)

answer:

A 3-sphere in Euclidean 4-space is defined by the equation

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Suppose we parameterize a curve on the 3-sphere by λ. Since we are in Euclidean 4-space, the infinitesimal distance corresponding to the vector $\frac{d}{d\lambda}$ is

$$dl^2 = [(\frac{d\vec{x}}{d\lambda})^2 + (\frac{dx_4}{d\lambda})^2]d\lambda^2$$

where $\vec{x} = (x_1, x_2, x_3)$. Since x_4 can be solved for in terms of $\vec{x}$, we write

$$x_4 = \pm\sqrt{R^2 - \vec{x}^2}.$$

Defining the usual spherical coordinates with the radial coordinate defined as $\tilde{r}$, we can rewrite this as

$$x_4 = \pm\sqrt{R^2 - \tilde{r}^2}.$$

Hence,

$$dx_4 = \frac{\pm\tilde{r}dr}{\sqrt{R^2 - \tilde{r}^2}}$$

and

$$dl^2 = d\vec{x}^2 + \frac{\tilde{r}^2 d\tilde{r}^2}{R^2 - \tilde{r}^2}.$$

Now, $d\vec{x}^2$ can be expressed in terms of usual spherical coordinates as

$$d\vec{x}^2 = d\tilde{r}^2 + \tilde{r}^2 d\Omega^2.$$

Hence, we find

$$\begin{aligned} dl^2 &= \frac{R^2 d\tilde{r}^2}{R^2 - \tilde{r}^2} + \tilde{r}^2 d\Omega^2 \\ &= \frac{d\tilde{r}^2}{1 - \frac{\tilde{r}^2}{R^2}} + \tilde{r}^2 d\Omega^2. \end{aligned}$$

As we have learned in class, the spatial part of the FRW metric for a fixed time t_0 is given by

$$d\vec{l}^2 = a^2(t_0)[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2]$$

which means that we can identify

$$a(t_0)dr = d\tilde{r}$$

$$K = \frac{a^2(t_0)}{R^2}.$$

We can always adjust $a(t_0)$ such that $K = 1$. Hence the spatial part of the FRW metric for $K = 1$ corresponds to the metric of a 3-sphere embedded in Euclidean 4-space.

3.: Because electrons combined with protons to form neutral hydrogen at temperature of about 0.3 eV, majority of the cosmic background photons that we see last scattered off of charged particles at that temperature. This is called the "last scattering surface." (All answers should be accurate at least to an order of magnitude.)

a): Given that the temperature of the cosmic background photon today is about 2.3×10^{-4}eV, compute the redshift z_* at the last scattering surface.

b): Compute the physical distance (on the homogeneous spacelike hypersurface today) from Earth to the last scattering surface. (i.e. The last scattering surface occupies a fixed coordinate distance with respect to us today. Compute the physical distance between that surface and us where the spatial metric is defined on the homogeneous spacelike hypersurface today.) Assume mattter domination.

c): How far can a photon travel from the time of big bang singularity to the time of the last scattering surface? If you call that distance X, a causally connected volume region at the time of last scattering surface is X^3 (assuming causal processes started at the big bang singularity).

d): About how many causally disconnected patches are there at the last scattering surface. [Use results of part c).]

answer:

(a):

As we have shown in class, redshift of light due to cosmological expansion is given as

$$(1+z(t)) = \frac{a(t_0)}{a(t)}$$

and the entropy conservation leads to

$$T \propto \frac{1}{a}.$$

Hence, we have

$$(1+z) = \frac{T}{T_0}.$$

Plugging in $T \sim 10^{-1}$ eV and $T_0 \sim 10^{-4}$eV, we find the redshift to be

$$z_l \sim 10^3$$

at the last scattering surface.

(b):

The physical distance is given as

$$l = a_0 \int dr.$$

To compute $\int dr$, we need to integrate the geodesic equation. The radial null geodesics are found by setting $ds^2 = 0$:

$$\begin{aligned} \int dr &= \int \frac{dt}{a} \\ &= \int \frac{da}{\dot{a}a} \\ &= \int \frac{da}{a^2\sqrt{\frac{8\pi}{3}\left[\rho + \frac{K}{a^2}\right]}} \end{aligned}$$

where we used the Friedmann equation in the last equality. Hence, we need to compute

$$\begin{aligned} l = a_0 \int_0^r dr &= \int_{a_{LSS}/a_0}^1 \frac{d(a/a_0)}{\sqrt{\frac{8\pi}{3}\left[\rho(\frac{a}{a_0})^4 + \frac{K}{a_0^2}(\frac{a}{a_0})^2\right]}} \\ &= \int_{\frac{1}{1+z_{LSS}}}^1 \frac{dx}{\sqrt{\frac{8\pi}{3}\left[\rho x^4 + \frac{K}{a_0^2}x^2\right]}} \end{aligned}$$

where $z_{LSS} \sim 10^3$ is the redshift to the last scattering surface. We need to specify ρ. Since

$$\rho_R/\rho_M|_{t_0} \sim 10^{-4}, \ \rho_M/\rho_{\text{dark energy}}|_{t_0} \sim \mathcal{O}(1)$$

and $K \approx 0$, and since $\rho_R/\rho_M \propto a \propto (1+z)$, we can to an order of magnitude accuracy simply assume matter domination to the z_{LSS}. Hence, we use

$$\rho \approx \rho_0 \left(\frac{a_0}{a}\right)^3$$

corresponding to matter domination and find

$$\begin{aligned} l &= \int_{\frac{1}{1+z_{LSS}}}^{1} \frac{dx}{\sqrt{\frac{8\pi}{3}\rho_0 x}} \\ &= \frac{1}{\sqrt{\frac{8\pi}{3}\rho_0}} \int_{\frac{1}{1+z_{LSS}}}^{1} \frac{dx}{\sqrt{x}} \\ &= \frac{2}{\sqrt{\frac{8\pi}{3}\rho_0}} \left(1 - \sqrt{\frac{1}{1+z_{LSS}}}\right) \end{aligned}$$

Substituting $z_{LSS} = 10^3$, we find

$$l = \frac{2}{\sqrt{\frac{8\pi}{3}\rho_0}}$$

Numerically, this corresponds to about $\frac{2}{70\text{km/s/Mpc}}(3 \times 10^5\text{km/s}) \approx 9 \times 10^3$ Mpc $\approx 3 \times 10^{28}$cm. That is a pretty large distance! The modern CMB measurements are seeing that far away!

(c):

As in part (b), using the null geodesic condition $ds^2 = 0$, we can write the length at the last scattering surface as

$$X = \frac{a_{LSS}}{a_0} \int_0^{\frac{1}{1+z_{LSS}}} \frac{dx}{\sqrt{\frac{8\pi}{3}\rho x^4}}$$

where as before we are neglecting the curvature part. Just as in part b), for an order of magnitude accuracy computation, neglect dark energy:

$$\rho \sim \rho_{R0} x^{-4} + \rho_{M0} x^{-3}.$$

We thus find

$$\begin{aligned} X &\sim \frac{1}{1+z_{LSS}} \int_0^{\frac{1}{1+z_{LSS}}} \frac{dx}{\sqrt{\frac{8\pi}{3}\left[\rho_{R0} + \rho_{M0} x\right]}} \\ &= \frac{1}{1+z_{LSS}} \int_0^{\frac{1}{1+z_{LSS}}} \frac{dx}{\sqrt{\frac{8\pi}{3}\rho_{M0}\left[\frac{\rho_{R0}}{\rho_{M0}} + x\right]}} \\ &= \frac{2}{\sqrt{\frac{8\pi}{3}\rho_{M0}}} \frac{1}{1+z_{LSS}} \left(\sqrt{\frac{\rho_{R0}}{\rho_{M0}} + \frac{1}{1+z_{LSS}}} - \sqrt{\frac{\rho_{R0}}{\rho_{M0}}}\right) \\ &\sim \frac{2}{\sqrt{\frac{8\pi}{3}\rho_{M0}}} \frac{1}{(1+z_{LSS})^{3/2}} \end{aligned}$$

Numerically, this is of the order of $\mathcal{O}(0.3)$ Mpc.

(d):

From the given question, the student could have written two different classes of answers: area patches and volume patches.

area patches: If the patch is interpreted as the area on the 2-surface of last scattering (surface of a 2-sphere surrounding earth), the student would work with the area observable to us (from part (b)), noting

$$\text{area today} = 4\pi l^2.$$

At the time of the last scattering surface, the area was

$$\begin{aligned} \text{area at the last scattering surface} &= 4\pi l^2 \left(\frac{a(t_{LSS})}{a(t_0)}\right)^2 \\ &= \frac{4\pi l^2}{(1+z_{LSS})^2} \end{aligned}$$

The number of causally connected patches at the last scattering surface is

$$\begin{aligned} N_c &= \text{area at the last scattering surface}/X^2 \\ &= \frac{4\pi l^2}{(1+z_{LSS})^2 X^2} \end{aligned}$$

where X was defined in part (c). Hence, in this case, the student would arrive at

$$\begin{aligned} N_c &\sim \mathcal{O}(10)\frac{\left\{\frac{2}{\sqrt{\frac{8\pi}{3}\rho_0}}\right\}^2}{(1+z_{LSS})^2\left\{\frac{2}{\sqrt{\frac{8\pi}{3}\rho_{M0}}}\frac{1}{(1+z_{LSS})^{3/2}}\right\}^2} \\ &\sim (1+z_{LSS})\mathcal{O}(10) \\ &\sim 10^4. \end{aligned}$$

volume patches: If the patch is interpreted as the volume on the 3-surface of last scattering, the student would work with volumes noting

$$\text{volume today bounded by last scattering surface} = \frac{4\pi}{3}l^3.$$

At the time of the last scattering surface, the same volume was scaled down by the scale factor

$$\begin{aligned} \text{volume at the last scattering surface} &= \frac{4\pi}{3}l^3(\frac{a(t_{LSS})}{a(t_0)})^3 \\ &= \frac{4\pi l^3}{3(1+z_{LSS})^3} \end{aligned}$$

The number of causally connected volumes within this volume is

$$\begin{aligned} N_c &= \text{volume at the last scattering surface}/X^3 \\ &\sim \frac{l^3}{(1+z_{LSS})^3 X^3} \end{aligned}$$

where X was defined in part (c). Hence, in this case, the student would arrive at

$$\begin{aligned} N_c &\sim \frac{\left\{\frac{2}{\sqrt{\frac{8\pi}{3}\rho_0}}\right\}^3}{(1+z_{LSS})^3\left\{\frac{2}{\sqrt{\frac{8\pi}{3}\rho_{M0}}}\frac{1}{(1+z_{LSS})^{3/2}}\right\}^3} \\ &\sim (1+z_{LSS})^{3/2} \\ &\sim \mathcal{O}(10^4) \end{aligned}$$

which is coincidentally similar.

3.: Find a solution to the Einstein's equations if the stress tensor is dominated by a cosmological constant. Write the solution in the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j$$

and find g_{ij}.

answer:

The cosmological constant is characterized by

$$\rho = \text{constant}.$$

Hence, Friedmann equation yields

$$\frac{\dot{a}}{a} \approx \frac{8\pi}{3}\sqrt{\rho}$$

which in turn yields

$$a \approx a_0 \exp\left(\frac{8\pi}{3}\sqrt{\rho}t\right).$$

Hence

$$g_{ij} \approx a_0 \exp\left(\frac{8\pi}{3}\sqrt{\rho}t\right)\delta_{ij}.$$

4.: Show that a self-gravitating over density relative to an average background density $\langle\rho\rangle$, i.e.

$$\delta(x) \equiv \frac{\rho(x) - \langle\rho\rangle}{\langle\rho\rangle}$$

will grow linearly with the expansion of a matter dominated universe.

answer:

In lecture, we saw

$$\Phi'' + 3\frac{a'}{a}\Phi' + \left[2\partial_\eta\left(\frac{a'}{a}\right) + (\frac{a'}{a})^2\right]\Phi = 4\pi a^2 \delta P \delta^i{}_j.$$

In matter dominated universe $P = 0 = \delta P$. Furthermore, since

$$\int \frac{dt}{a} = \int d\eta$$

and since during matter domination, $a \propto t^{2/3}$ as we saw in class, we find

$$t^{1/3} \propto \eta$$

or

$$a = a_0\left(\frac{\eta}{\eta_0}\right)^2.$$

Hence,

$$2\partial_\eta\left(\frac{a'}{a}\right) + (\frac{a'}{a})^2 = 0$$

and the equation governing the potential becomes simple:

$$\Phi'' + \frac{6}{\eta}\Phi' = 0.$$

Just as in lecture, separation of variables with each mode labeled by k allows us to trivially integrate to find the time dependence of each mode

$$\Phi = \sum_k (c_1(k) + \frac{c_2(k)}{\eta^5}) f_k(\vec{x}).$$

Since the $\delta\rho$ equation is

$$-3\frac{a'}{a}(\frac{a'}{a}\Psi + \Phi') + \partial_i^2\Phi = 4\pi a^2\delta\rho$$

we find that as $\eta \to \infty$

$$\delta\rho \sim \frac{1}{4\pi a^2}\sum_k c_1(k)\partial_i^2 f_k(\vec{x})$$

since all other terms fall off faster with time. Since $\langle\rho\rangle \propto a^{-3}$, we have arrived at

$$\frac{\delta\rho}{\rho} \propto a$$

as $\eta \to \infty$.