PHYSICS 717 PROBLEM SET 11

due: Monday, April 20, 2009, at the beginning of lecture

Problems

1.: Derive starting with

$$G_{ab} = 8\pi T_{ab}$$

and

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

the linearized equation

$$-\frac{1}{2}\partial_\lambda \partial^\lambda \bar{h}_{\alpha\beta} + \partial^\lambda \partial_{(\beta} \bar{h}_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda \partial^\gamma \bar{h}_{\lambda\gamma} = 8\pi T_{\alpha\beta}$$

where

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h.$$

(Hint:: Evaluate the Ricci tensor and Ricci scalar in the frame in which $\Gamma^\lambda_{\alpha\beta} = 0$.)

answer:

To linear order in $h_{\alpha\beta}$ and the definition of $\Gamma^\gamma_{\alpha\beta}$, it is obvious that

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\lambda}(h_{\alpha\lambda,\beta} + h_{\beta\lambda,\alpha} - h_{\alpha\beta,\lambda})$$

since the h term in the inverse metric contributes higher order in h perturbations. In the frame in which $\Gamma^\lambda_{\alpha\beta} = 0$, we find

$$\begin{aligned} R_{\alpha\beta} &= \Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\lambda\beta,\alpha} \\ &= \frac{1}{2}\partial^\lambda(h_{\alpha\lambda,\beta} + h_{\beta\lambda,\alpha} - h_{\alpha\beta,\lambda}) - \frac{1}{2}\eta^{\gamma\lambda}\partial_\alpha(h_{\gamma\lambda,\beta} + h_{\beta\lambda,\gamma} - h_{\gamma\beta,\lambda}) \\ &= \partial^\lambda\partial_{(\beta}h_{\alpha)\lambda} - \frac{1}{2}\partial^\lambda\partial_\lambda h_{\alpha\beta} - \frac{1}{2}h_{,\beta\alpha} \end{aligned}$$

Finally, we can write

$$R = \partial^\lambda\partial^\beta h_{\lambda\beta} - \partial^\lambda\partial_\lambda h.$$

Hence

$$G_{\alpha\beta} = \partial^\lambda\partial_{(\beta}h_{\alpha)\lambda} - \frac{1}{2}\partial^\lambda\partial_\lambda h_{\alpha\beta} - \frac{1}{2}h_{,\beta\alpha} - \frac{1}{2}\eta_{\alpha\beta}\left[\partial^\lambda\partial^\beta h_{\lambda\beta} - \partial^\lambda\partial_\lambda h\right].$$

To put this in a neater form, define

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$$

and evaluate

$$\begin{aligned} \frac{-1}{2}\partial^\lambda\partial_\lambda\bar{h}_{\alpha\beta} + \partial^\lambda\partial_{(\beta}\bar{h}_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda\partial^\gamma\bar{h}_{\lambda\gamma} &= \\ \frac{-1}{2}\partial^\lambda\partial_\lambda h_{\alpha\beta} + \partial^\lambda\partial_{(\beta}h_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda\partial^\gamma h_{\lambda\gamma} & \\ \frac{1}{4}\partial^\lambda\partial_\lambda(\eta_{\alpha\beta}h) - \frac{1}{4}\partial_\alpha\partial_\beta h - \frac{1}{4}\partial_\beta\partial_\alpha h + \frac{1}{2}\eta_{\alpha\beta}\frac{1}{2}\eta_{\lambda\gamma}\partial^\lambda\partial^\gamma h &= \\ \frac{-1}{2}\partial^\lambda\partial_\lambda h_{\alpha\beta} + \partial^\lambda\partial_{(\beta}h_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda\partial^\gamma h_{\lambda\gamma} & \\ -\frac{1}{2}\partial_\alpha\partial_\beta h + \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda\partial_\lambda h & \end{aligned}$$

Hence, we have verified

$$-\frac{1}{2}\partial_\lambda \partial^\lambda \bar{h}_{\alpha\beta} + \partial^\lambda \partial_{(\beta} \bar{h}_{\alpha)\lambda} - \frac{1}{2}\eta_{\alpha\beta}\partial^\lambda \partial^\gamma \bar{h}_{\lambda\gamma} = 8\pi T_{\alpha\beta}$$

2.: Recall from lecture that the Lorentz gauge residual gauge fixing conditions $h = 0$ and $h_{\mu 0} = 0$ can both be satisfied simultaneously. Check this explicitly by showing that

$$\left\{ A_{00} + 2i\beta_0 k_0 = \frac{-1}{2}A^\alpha{}_\alpha,\ i\left[\beta_i k_0 + \beta_0 k_i\right] = -A_{0i} \right\}$$

$$i\beta^\gamma k_\gamma = \frac{1}{2} A^\alpha{}_\alpha$$

which is equivalent to $h = 0$. (The definition of the notation is from lecture.)

answer:

The equation $A_{00} + 2i\beta_0 k_0 = \frac{-1}{2} A^\alpha{}_\alpha$, can be solved for β_0 to obtain

$$\beta_0 = \frac{(-A_{00} - \frac{1}{2} A^\alpha{}_\alpha)}{2ik_0}$$

Solving $i\left[\beta_i k_0 + \beta_0 k_i\right] = -A_{0i}$ for β_i then gives

$$\beta_i = \left[iA_{0i} - \frac{(-A_{00} - \frac{1}{2} A^\alpha{}_\alpha)}{2ik_0} k_i \right] \frac{1}{k_0}.$$

Hence, we have

$$\begin{aligned} i\beta^\gamma k_\gamma &= i\left[\beta^0 k_0 + \beta^i k_i\right] \\ &= \frac{(A_{00} + \frac{1}{2} A^\alpha{}_\alpha)}{2} + \left[-A_{0i} - \frac{(-A_{00} - \frac{1}{2} A^\alpha{}_\alpha)}{2k_0} k_i \right] \frac{k_i}{k_0} \\ &= \frac{(A_{00} + \frac{1}{2} A^\alpha{}_\alpha)}{2} + -A_{0i} \frac{k_i}{k_0} - \frac{(-A_{00} - \frac{1}{2} A^\alpha{}_\alpha)}{2} \end{aligned}$$

where the last equality is obtained by using the dispersion relationship $k_i k_i = k_0^2$. Hence

$$i\beta^\gamma k_\gamma = -A_{0i}\frac{k_i}{k_0} + A_{00} + \frac{1}{2} A^\alpha{}_\alpha$$

Since $\partial_\mu \bar{h}^{\mu\nu} = 0$,

$$ik_0 A_{00} = ik_i A_{i0}.$$

Hence, we find

$$\begin{aligned} i\beta^\gamma k_\gamma &= -A_{00} + A_{00} + \frac{1}{2} A^\alpha{}_\alpha \\ &= \frac{1}{2} A^\alpha{}_\alpha. \end{aligned}$$

3.: Green's function exercise: Suppose you are told that the gravitational waves in a strange world obey the following equation:

$$\partial^\lambda \partial_\lambda \bar{h}_{\alpha\beta}(t, \vec{x}) - m^2 \bar{h}_{\alpha\beta}(t, \vec{x}) = -16\pi T_{\alpha\beta}$$

with an external source given by

$$T_{\alpha\beta} = M\Theta(t)\delta^{(3)}(x)\delta_{\alpha 0}\delta_{\beta 0}$$

where $\{\omega,\ R,\ m\}$ are positive constants and $\omega L \ll 1$. ($\Theta(t)$ is a step function which is unity when $t > 0$ and zero otherwise). Compute the time average

$$\langle \bar{h}_{\alpha\beta} \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \bar{h}_{\alpha\beta}(t, \vec{x})$$

for the solution satisfying boundary conditions $\bar{h}_{\alpha\beta}|_{t=-1/m} = 0$ and $\partial_t \bar{h}_{\alpha\beta}|_{t=-1/m} = 0$. You may find the following integral useful:

$$\int_{-\infty}^{\infty} dy \frac{y \sin(yc)}{1 + y^2} = \pi e^{-c}$$

for $c > 0$.

answer:

By the method of Green's functions, one can immediately write down the formal answer

$$\bar{h}_{\alpha\beta} = -16\pi \int d^4x' G(x, x') T_{\alpha\beta}(x')$$

up to boundary conditions which we can fulfill by adding homogeneous solutions if necessary. To find $G(x, x')$, we take the Fourier transform as in the lecture notes:

$$\tilde{G}(k) = \frac{-1}{k^\mu k_\mu + m^2}.$$

Writing the inverse Fourier transform, we find

$$G(x, x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{-(k^0)^2 + \vec{k}^2 + m^2}$$

for $m = 0$. Doing the integral, we have $\int d^4x' G(x,x') T_{\alpha\beta}(x')$ being

$$\begin{aligned}
\bar{h}_{\alpha\beta} &= -16\pi \int d^4x' G(x,x') M\Theta(t')\delta^{(3)}(x')\delta_{\alpha 0}\delta_{\beta 0} \\
&= -16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^\infty dt' G(t,\vec{x};t',0) \\
&= 16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^\infty dt' \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\vec{k}\cdot\vec{x}} e^{-ik^0(t-t')}}{-(k^0)^2 + \vec{k}^2 + m^2}.
\end{aligned}$$

Integrating over dk^0, we find

$$I \equiv \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\vec{k}\cdot\vec{x}} e^{-ik^0(t-t')}}{-(k^0)^2 + \vec{k}^2 + m^2} = -\int \frac{d^3k dk^0}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} \frac{e^{-ik^0(t-t')}}{(k^0 - (w_k - i\epsilon))(k^0 - (-w_k - i\epsilon))}$$

where the $i\epsilon$ enforces causal boundary condition for the Green's function and $w_k \equiv \sqrt{m^2 + \vec{k}^2}$. Carrying out the integral

$$\begin{aligned}
I &= -\int \frac{d^3k}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} \Theta(t-t')(-2\pi i) \left\{ \frac{e^{-iw_k(t-t')}}{2w_k} - \frac{e^{iw_k(t-t')}}{2w_k} \right\} \\
&= -\Theta(t-t') \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\sin[w_k(t'-t)]}{w_k} \\
&= -\Theta(t-t') \int_0^\infty \frac{dk k^2 d\cos\theta}{(2\pi)^2} e^{ikr\cos\theta} \frac{\sin[w_k(t'-t)]}{w_k} \\
&= -\Theta(t-t') \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{e^{ikr} - e^{-ikr}}{ikr} \right] \frac{\sin[w_k(t'-t)]}{w_k} \\
&= -\Theta(t-t') \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \frac{\sin[w_k(t'-t)]}{w_k}
\end{aligned}$$

Hence, the solution is

$$\begin{aligned}
\bar{h}_{\alpha\beta} &= -16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^\infty dt' \Theta(t-t') \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \frac{\sin[w_k(t'-t)]}{w_k} \\
&= -16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^t dt' \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \frac{\sin[w_k(t'-t)]}{w_k} \\
&= -16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \left(\frac{-1 + \cos[w_k t]}{w_k^2} \right) \Theta(t)
\end{aligned}$$

Clearly, this satisfies the boundary condition at $t < 0$. Time averaging, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \cos[w_k t] = \lim_{T\to\infty} \frac{\sin[w_k T]}{w_k T} = 0$$

$$\begin{aligned}
\langle \bar{h}_{\alpha\beta} \rangle &= 16\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_0^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \frac{1}{w_k^2} \\
&= 8\pi\delta_{\alpha 0}\delta_{\beta 0} M \int_{-\infty}^\infty \frac{dk k^2}{(2\pi)^2} \left[\frac{2\sin kr}{kr} \right] \frac{1}{k^2 + m^2} \\
&= 8\pi\delta_{\alpha 0}\delta_{\beta 0} M m \int_{-\infty}^\infty \frac{dy y^2}{(2\pi)^2} \left[\frac{2\sin yc}{yc} \right] \frac{1}{y^2 + 1} \\
&= 8\pi\delta_{\alpha 0}\delta_{\beta 0} \frac{Mm}{2\pi^2} \frac{e^{-c}\pi}{c} \\
&= 4\delta_{\alpha 0}\delta_{\beta 0} \frac{M}{r} e^{-mr}
\end{aligned}$$

where in the intermediate steps, we used $c \equiv rm$.

4.: Calculate the gravitational radiation luminosity of a spining thin uniform metal rod of mass M and length l, spinning at frequency w around a symmetrical perpendicular axis. Estimate the electromagnetic luminosity which would arise from the slight excess of electrons pushed toward the ends by "centrifugal force." (Note the electromagnetic radiation is approximately quadrupole in this case since the dipole charge distribution is approximately zero.) If the rod has a reasonable density (10g/cm^3) and is rotating at a reasonable frequency (1 kHz), will electromagnetic or gravitational radiation be more important in slowing the rotation?

answer:

from the formula

$$P = \frac{1}{45}\sum_{ij=1}^{3}(\frac{d^3Q_{ij}}{dt^3}(t-R))^2$$

where

$$Q_{\mu\nu} = q_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu}q$$

$$q_{\mu\nu} = 3\int T^{00}(t,\vec{x})x^\mu x^\nu d^3x$$

where $i,j \in \{1,23\}$. The energy density of the rod when not spinning and lying along the x-axis is

$$\tilde{T}^{00}(t,r) = \frac{M}{l}\delta(\tilde{y})\delta(\tilde{z})\theta(\frac{l}{2}-\tilde{x})\theta(\tilde{x}+\frac{l}{2})$$

where θ is the step function. Assuming nonrelativistic rotation speeds, we can treat q^{ij} like a 3-tensor and make the rotation transformation after computing in the rest frame. In the rest frame, we can simply compute

$$\begin{aligned} q^{ij} &= 3\int d^3\tilde{x}\tilde{x}^i\tilde{x}^j\delta(\tilde{y})\delta(\tilde{z})\theta(\frac{l}{2}-\tilde{x})\theta(\tilde{x}+\frac{l}{2})\frac{M}{l} \\ &= \frac{3M}{l}\delta^{i1}\delta^{j1}\frac{\tilde{x}^3}{3}|_{-l/2}^{l/2} \\ &= \frac{Ml^2}{4}\delta^{i1}\delta^{j1}. \end{aligned}$$

When spinning, assuming nonrelativistic rotation speeds, we can write

$$q^{ij} = \frac{\partial x^i}{\partial\tilde{x}^k}\frac{\partial x^j}{\partial\tilde{x}^l}\tilde{q}^{kl}$$

with the coordinate transformations given by

$$\begin{pmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, for $i,j \in \{1,2\}$, we have

$$\begin{aligned} q^{ij} &= \frac{Ml^2}{4}\begin{pmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{Ml^2}{4}\begin{pmatrix} \cos^2 wt & -\sin wt\cos wt \\ -\sin wt\cos wt & \sin^2 wt \end{pmatrix} \end{aligned}$$

and

$$q^{33} = q^{13} = q^{23} = 0.$$

$$\begin{aligned} Q_{11} &= \frac{Ml^2}{4}[\cos^2 wt - \frac{1}{3}] \\ &= \frac{Ml^2}{4}[\frac{1}{2}\cos 2wt + \frac{1}{2} - \frac{1}{3}] \end{aligned}$$

$$\begin{aligned} Q_{22} &= \frac{Ml^2}{4}[\sin^2 wt - \frac{1}{3}] \\ &= \frac{Ml^2}{4}[\frac{1}{2} - \frac{1}{2}\cos 2wt - \frac{1}{3}] \end{aligned}$$

$$Q_{33} = \frac{-1}{3}\frac{Ml^2}{4}$$

$$Q_{12} = q^{12} = -\frac{Ml^2}{4}\sin wt\cos wt = -\frac{Ml^2}{4}\frac{1}{2}\sin 2wt$$

$$Q_{13} = 0 = Q_{23}$$

$$
\begin{aligned}
\langle P \rangle &= \frac{1}{45}\sum_{ij=1}^{3}\left(\frac{d^3Q_{ij}}{dt^3}(t-R)\right)^2 = \frac{1}{45}[\langle \dddot{Q}_{11}\rangle^2 + \langle \dddot{Q}_{22}\rangle^2 + 2\langle \dddot{Q}_{12}\rangle^2 + \text{time independent terms}] \\
&= \frac{1}{45}\left(\frac{Ml^2}{4}\right)^2[\langle\{\frac{1}{2}(2w)^3\sin 2wt\}^2\rangle + \langle\{-\frac{1}{2}(2w)^3\sin 2wt\}^2\rangle + 2\langle\{\frac{1}{2}(2w)^3\cos 2wt\}^2\rangle] \\
&= \frac{1}{45}\frac{M^2l^4}{16}[16w^6\frac{1}{2} + 16w^6\frac{1}{2} + 2(16w^6\frac{1}{2})] \\
&= \frac{2}{45}M^2l^4w^6
\end{aligned}
$$

We would now like to compare this with electromagnetic radiation. As the bar is rotating, electrons will be pushed against the end, producing a charge configuration of $- + + -$. If the electric potential is given by ϕ, we have at radial location r

$$|e\nabla\phi| = rm_e w^2$$

where m_e is the mass of the electron. This implies that the charge distribution $|\rho_e| = |\frac{1}{4\pi}\nabla^2\phi| \sim \frac{1}{4\pi}\frac{m_e w^2}{e}$. Note that the dipole moment $\vec{d}$ of this charge distribution vanishes:

$$
\begin{aligned}
|\vec{d}| &= |\int \rho_e(x)\vec{x}d^3x| \\
&\sim |\frac{1}{4\pi}\int_{-l/2}^{l/2}\frac{m_e w^2}{e}xdxA| = 0
\end{aligned}
$$

where A is the cross sectional area. This vanishes since x is an odd function and the integral is symmetric. Hence, just as in the gravitational radiation, the leading radiation contribution is the quadrupole and not the dipole.

The electric quadrupole moment is

$$d^{ij} = 3\int \rho_e x^i x^j d^3x$$

which can be estimated as

$$d^{11} \sim \frac{m_e w^2}{e} l^3 A\cos^2 wt.$$

Note that it does not vanish because x^2 is an even function. This should yield, just as for the gravitational radiation

$$\langle P_{em}\rangle \sim \langle(\dddot{d}^{11})^2\rangle \sim \left(\frac{m_e w^5}{e} l^3 A\right)^2.$$

The ratio of this to the gravity wave power is

$$\frac{\langle P_{em}\rangle}{\langle P\rangle} \sim \frac{m_e^2 w^{10} l^6 A^2}{e^2}\frac{1}{M^2 l^4 w^6} = \frac{m_e^2 w^4 l^2 A^2}{e^2 M^2} = \frac{m_e^2 w^4}{e^2\rho^2}.$$

Restoring G_N, we have

$$\frac{\langle P_{em}\rangle}{\langle P\rangle} \sim \frac{m_e^2 w^4}{e^2\rho^2 G_N}$$

Using

$$
\begin{aligned}
\frac{m_e}{e} &\sim \frac{0.511\text{ MeV}}{4.8\times 10^{-10}\text{esu}} = \frac{0.511\text{ MeV}}{0.3} \\
G_N &\sim \frac{1}{(1.22\times 10^{19}\text{GeV})^2} \\
\rho &\sim 10\text{g/cm}^3 \sim 4\times 10^{-17}\text{GeV}^4 \\
w &\sim 2\pi\text{kHz} \sim 4\times 10^{-21}\text{GeV} \\
\frac{\langle P_{em}\rangle}{\langle P\rangle} &\sim 10^{-17}.
\end{aligned}
$$

Amazingly enough, the gravitational radiation wins over the electromagnetic radiation. Of course, this is a rare situation since we had the dipole vanishing and the quadrupole induced very weakly by rotational dynamics.