PROBLEM SET 1 SOL

due: Monday, Feb 2, 2009, at the beginning of lecture

Problems

(1) Consider the choice of units where $G_N = \hbar = 1$ but not c. Recall that in the SI units, we have

$$[\hbar] = kg\, m^2/s$$

$$[G_N] = m^3/kg/s^2$$

(a) Write down the time dependent nonrelativistic Schroedinger equation (in the spacetime position basis) for a point particle wave function $\psi(t, \vec{x})$ of mass m in the presence of a central gravitational potential due to a point mass $M \gg m$ fixed at the origin.

answer

$$\frac{-1}{2m}\vec{\nabla}^2\psi - \frac{mM}{r}\psi = i\partial_t\psi$$

(b) What is the power n for the units distance^n describing energy?

answer

Energy has units of kg m^2/s^2 in SI. Since I must recover this unit by multiplying by appropriate $[\hbar]^x$ and $[G_N]^y$ to distance^n, I write the equation

$$1 = x - y$$

$$2 = 2x + 3y + n$$

$$-2 = -x - 2y$$

One can easily solve for n to be

$$n = -5/3.$$

(c) Suppose you are given that one eigenenergy of the system has the value E_0 in the present units. What numerical value must be multiplied to E_0 to conver the energy value to a Joule?

answer

Since $x = 4/3$ and $y = 1/3$ in the previous part, we find

$$E_{(joule)} = E_0\hbar^x G_N^y = E_0(1.1 \times 10^{-34})^{4/3}(6.7 \times 10^{-11})^{1/3}.$$

Hence the numerical factor is 2×10^{-49}, assuming lengths were originally measured in meters.

(2) It is easy to see that

$$\epsilon_{ijk}\partial_j A_k = (\vec{\nabla} \times \vec{A})_i.$$

(a) Prove the identity

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

answer

Start with the first case

$$\epsilon_{12k}\epsilon_{12k} = \epsilon_{123}\epsilon_{123} = 1.$$

Hence, we might guess

$$\epsilon_{ijk}\epsilon_{lmk} \sim \delta_{il}\delta_{jm}$$

However, this term by itself does not become negative under exchange of the indices ij. Hence, antisymmetrizing in ij, we find

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

which is also automatically antisymmeterized in lm exchange.

(b) Use the index notation to show that

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}$$

answer

$$\begin{aligned}\epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l A_m &= \epsilon_{ijk}\partial_j\epsilon_{lmk}\partial_l A_m\\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial_j\partial_l A_m\\ &= \partial_i\vec{\nabla}\cdot\vec{A}-\vec{\nabla}^2 A_i\end{aligned}$$

(3) Verify that the proper homogeneous Lorentz boosts

$$\Lambda^i{}_j=\delta_{ij}+v_i v_j\frac{(\gamma-1)}{\vec{v}^2}$$

$$\Lambda^0{}_j=\gamma v_j$$

satisfy

$$\eta_{\lambda\gamma}=\eta_{\alpha\beta}\Lambda^\alpha{}_\lambda\Lambda^\beta{}_\gamma.$$

answer

$$Q_{\lambda\gamma}\equiv\eta_{\alpha\beta}\Lambda^\alpha{}_\lambda\Lambda^\beta{}_\gamma=\eta_{00}\Lambda^0{}_\lambda\Lambda^0{}_\gamma+\eta_{ij}\Lambda^i{}_\lambda\Lambda^j{}_\gamma$$

Since $Q_{\lambda\gamma}=Q_{\gamma\lambda}$ and since we already worked out Q_{00}in class, we need to only work out the 9 remaining independent components. Start with

$$\begin{aligned}Q_{ln} &= \eta_{00}\Lambda^0{}_l\Lambda^0{}_n+\eta_{ij}\Lambda^i{}_l\Lambda^j{}_n\\ &= -\gamma^2 v_l v_n+\delta_{ij}(\delta_{il}+v_i v_l\frac{\gamma-1}{v^2})(\delta_{jn}+v_j v_n\frac{\gamma-1}{v^2})\\ &= -\gamma^2 v_l v_n+(\delta_{nl}+v_n v_l\frac{\gamma-1}{v^2}+v_l v_n\frac{\gamma-1}{v^2}+v_l v_n\frac{(\gamma-1)^2}{v^2})\\ &= -\gamma^2 v_l v_n+(\delta_{nl}+2v_n v_l\frac{\gamma-1}{v^2}+v_l v_n\frac{(\gamma^2+1-2\gamma)}{v^2})\\ &= -\gamma^2 v_l v_n+(\delta_{nl}+2v_n v_l\frac{-1}{v^2}+v_l v_n\frac{(\gamma^2+1)}{v^2})\\ &= \delta_{nl}+v_n v_l[\frac{-1}{v^2}+\frac{\gamma^2}{v^2}-\gamma^2]\end{aligned}$$

Now, using

$$\gamma^2-1=\frac{1}{1-v^2}-1=\frac{1-(1-v^2)}{1-v^2}=\frac{v^2}{1-v^2}=\gamma^2 v^2,$$

we find

$$Q_{ln}=\delta_{nl}$$

Finally, we write

$$\begin{aligned}Q_{l0} &= \eta_{00}\Lambda^0{}_l\Lambda^0{}_0+\eta_{ij}\Lambda^i{}_l\Lambda^j{}_0\\ &= -v_l\gamma^2+\delta_{ij}(\delta_{il}+v_i v_l\frac{\gamma-1}{v^2})\gamma v_j\\ &= -v_l\gamma^2+\gamma v_l+v_l\gamma(\gamma-1)=0\end{aligned}$$

Hence, rest of the 9 components were confirmed.