PHYSICS 717 PROBLEM SET 2 SOLUTIONS

due: Monday, Feb 9, 2009, at the beginning of lecture

Problems

1.: Lesson in signs: Show that the $\Lambda^\mu{}_\nu(\vec{v})$ in problem 3 of the last homework has the interpretation of boosting particle velocities (making objects at rest move at speed $\vec{v}$). Such boosts are sometimes referred to as active boost transformations, as the substitution

$$dx^\mu \to dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$$

changes the state of the system. Show that from a coordinate change perspective, $\Lambda^\mu{}_\nu(\vec{v})$ defined according to

$$dx'^\mu = \Lambda^\mu{}_\nu(\vec{v}) dx^\nu = \begin{pmatrix} \gamma & -\gamma\vec{v} \\ -\gamma\vec{v} & \delta^{ij} + v^i v^j \frac{\gamma-1}{|\vec{v}|^2} \end{pmatrix} \begin{pmatrix} dt \\ d\vec{x} \end{pmatrix} \tag{0.1}$$

has the interpretation of passive transformation, in which there is simply a coordinate transformation between old coordinates and new coordinates in which the new coordinate origin is moving at velocity $\vec{v}$ with respect to the old coordinate origin. For example, what is the 3-velocity of a particle at rest transformed according to Eq. (0.1)?

answer:

Start with $d\vec{x}/dt = (0,0,0)$ and consider the same motion after a coordinate transformation to the $(t', \vec{x}')$ frame moving at $\vec{v}$ with respect to the old coordinate system.

$$\begin{aligned} dx'^i &= \Lambda^i{}_\mu dx^\mu = \Lambda^i{}_0 dt + \Lambda^i{}_j dx^j \\ &= -\gamma v^i dt. \end{aligned}$$

The time coordinate is

$$\begin{aligned} dt' &= \Lambda^0{}_0 dt + \Lambda^0{}_j dx^j \\ &= \gamma dt. \end{aligned}$$

Hence,

$$\frac{d\vec{x}'}{dt'} = -\vec{v}$$

which is what we expect of the particle in the new coordinate system moving with speed $\vec{v}$ with respect to the old coordinate system.

2.: If two frames move with 3-velocities $\vec{u}$ and $\vec{w}$, show that their relative velocity magnitude is given by

$$\vec{v}^2 = \frac{(\vec{u}-\vec{w})^2 - (\vec{u}\times\vec{w})^2}{(1-\vec{u}\cdot\vec{w})^2}$$

answer:

Let u_1^α and u_2^α be 4-velocities of the two frames. In frame 1, $u_1^\alpha = (1,0)$, $u_2^\beta = (\gamma, \gamma\vec{v})$. Now, note that

$$u_1^\alpha u_2^\beta \eta_{\alpha\beta} = -\gamma$$

which is a Lorentz invariant. In a general coordinate system,

$$u_1^\alpha = (\gamma_1, \gamma_1 \vec{v}_1)$$

$$u_2^\beta = (\gamma_2, \gamma_2 \vec{v}_2)$$

where we have called $\vec{u} = \vec{v}_1$ and $\vec{w} = \vec{v}_2$ in the original statement of the problem. Hence,

$$\gamma = -u_1^\alpha u_2^\beta \eta_{\alpha\beta} = \gamma_1\gamma_2 - \gamma_1\gamma_2 \vec{v}_1\cdot\vec{v}_2$$

Therefore,

$$\sqrt{1-v^2} = \frac{1}{\gamma_1\gamma_2(1-\vec{v}_1\cdot\vec{v}_2)}$$

or

$$1 - v^2 = \frac{(1-v_1^2)(1-v_2)^2}{(1-\vec{v}_1\cdot\vec{v}_2)^2}$$

which gives

$$\begin{aligned} v^2 &= \frac{(1-\vec{v}_1\cdot\vec{v}_2)^2-(1-v_1^2)(1-v_2^2)}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \\ &= \frac{1+|\vec{v}_1|^2|\vec{v}_2|^2\cos^2\theta-2|\vec{v}_1||\vec{v}_2|\cos\theta-(1-v_1^2-v_2^2+v_1^2v_2^2)}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \\ &= \frac{|\vec{v}_1|^2|\vec{v}_2|^2\cos^2\theta-2|\vec{v}_1||\vec{v}_2|\cos\theta+(v_1^2+v_2^2-v_1^2v_2^2)}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \\ &= \frac{v_1^2v_2^2(\cos^2\theta-1)-2v_1v_2\cos\theta+(v_1^2+v_2^2)}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \\ &= \frac{-|\vec{v}_1\times\vec{v}_2|^2-2v_1v_2\cos\theta+(v_1^2+v_2^2)}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \\ &= \frac{(\vec{v}_1-\vec{v}_2)^2-|\vec{v}_1\times\vec{v}_2|^2}{(1-\vec{v}_1\cdot\vec{v}_2)^2} \end{aligned}$$

Thus, identifying $\vec{w}=\vec{v}_1$ and $\vec{u}=\vec{v}_2$, we find

$$v^2=\frac{(\vec{w}-\vec{u})^2-|\vec{w}\times\vec{u}|^2}{(1-\vec{w}\cdot\vec{u})^2}.$$

3.: What is the threshold kinetic energy K_{th} of the incident electron for the following process to occur? electron (fast) + proton (at rest) → electron + antiproton + two protons? Express your answer in terms of the masses of proton (= mass of antiproton) and electron.

answer:

The 4-momentum conservation is

$$p_1+P_2=p_3+P_4+P_5+P_6$$

in an obvious notation where $\{P_2,P_4,P_5,P_6\}$ correspond to (anti)proton momenta and P_2 is the momentum of the initial state proton at rest. Squaring both sides in a Lorentz invariant way, we have

$$(p_1+P_2)^2=(p_3+P_4+P_5+P_6)^2.$$

The left hand side can be evaluated in the lab frame while the right hand side is evaluated in the CM frame since the quantities on both sides are Lorentz invairant. We thus find

$$m_e^2+m_p^2+2E_em_p=(m_e+3m_p)^2.$$

Hence,

$$\begin{aligned} E_e &= \frac{1}{2m_p}\left[(m_e+3m_p)^2-m_e^2-m_p^2\right] \\ &= 4m_p+3m_e. \end{aligned}$$

The threshold kinetic energy is

$$K_{th}=E_e-m_e=4m_p+2m_e.$$

4.: Exercises with the Field strength tensor.

a): Show that for an observer moving with a 4-velocity u^a, the quantity $E_a\equiv F_{ab}u^b$ is interpreted as the electric field and $B_a=\frac{-1}{2}\epsilon_{ab}{}^{cd}F_{cd}u^b$ is interpreted as the magnetic field.

answer:

a) Note that the electric and magnetic fields are in general not tensors. However, one can define a tensor quantity

$$E_a\equiv F_{ab}u^b$$

which can be interpreted as the electric field from the reference point of an observer moving with 4-velocity u^a. This can be seen directly by going to the restframe of the observer in which $u^b=(1,0,0,0)$. In this frame, we have $E_\mu=F_{\mu 0}$. Lowering the indices, we have

$$F_{\mu\nu}=\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

which gives $E_0=0$ and $E_i=E^i$ for $i\in\{1,2,3\}$ where the right hand side E^i are the usual electric fields which we know are not tensors. Now given that $E_a\equiv F_{ab}u^b$ is a tensor equation, this equation is true in all frames of reference as the electric field seen by a local observer moving with velocity u^a. You can easily see that this generalizes to many other frames of reference.

b): Suppose a given $F^{\mu\nu}$ in frame O is observed in a frame $\bar{O}$ which is moving with a velocity $v\hat{x}$ with respect to O. Evaluate explicitly the matrix components of $\bar{F}^{\mu\nu}$ in terms of E^i, B^i, v, and γ.

answer:

Under Lorentz transformations with a boost along the $\hat{x}$ axis, we can write

$$\bar{F}^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$$

where

$$\Lambda^\mu{}_\alpha = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

For example, this matrix maps the worldline $(t, 0, 0, 0)$ into $(\gamma t, -\gamma v t, 0, 0)$. Multiplying this out, we easily find

$$\bar{F}^{\mu\nu} = \begin{pmatrix} 0 & E^1 & \gamma(E^2 - B^3 v) & \gamma(E^3 + B^2 v) \\ -E^1 & 0 & \gamma(B^3 - E^2 v) & -\gamma(B^2 + E^3 v) \\ -\gamma(E^2 - B^3 v) & -\gamma(B^3 - E^2 v) & 0 & B^1 \\ -\gamma(E^3 + B^2 v) & \gamma(B^2 + E^3 v) & -B^1 & 0 \end{pmatrix}$$

c): Show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

answer:

Consider

$$F_{0i} = \partial_0 A_i - \partial_i A_0$$

which gives the usual expression for the electric field. While the rest

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = B_3$$

$$F_{13} = \partial_1 A_3 - \partial_3 A_1 = -B_2$$

$$F_{23} = \partial_2 A_3 - \partial_3 A_2 = B_1$$

are the usual expressions for the $\vec{B}$ field. Note that this form of an antisymmetric $(0, 2)$ tensor (or 2-form) automatically satisfies the Maxwell's equations

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0$$

because of the antisymmetric nature of the ϵ tensor.

5.: Show that

$$f^\alpha = eF^\alpha{}_\gamma \frac{dx^\gamma}{d\tau}$$

and

$$\frac{dp^\alpha}{d\tau} = f^\alpha$$

expresses the Lorentz force law

$$\frac{d\vec{p}}{dt} = e[\vec{E} + \vec{v} \times \vec{B}].$$

answer:

Start with the expression

$$f^\alpha = eF^\alpha{}_\gamma \frac{dx^\gamma}{d\tau}$$

whose spatial component implies

$$\frac{dp^j}{d\tau} = -eF^{j0}\frac{dx^0}{d\tau} + e\sum_{i=1}^{3} F^{ji}\frac{dx^i}{d\tau}$$

By multiplying both sides by $\frac{d\tau}{dx^0}$, the second of thes equations can be written as

$$\frac{dp^j}{dt} \quad = \quad -eF^{j0} + e\sum_{i=1}^{3} F^{ji}\frac{dx^i}{dt}.$$

Now, writing out the components,

$$\begin{aligned}
\frac{dp^1}{dt} &= eE^1 + e\sum_{i=1}^{3} F^{1i}\frac{dx^i}{dt} \\
&= eE^1 + e[B^3\frac{dx^2}{dt} - B^2\frac{dx^3}{dt}] \\
&= eE^1 + e(\vec{v}\times\vec{B})^1 \\
\frac{dp^2}{dt} &= eE^2 + e\sum_{i=1}^{3} F^{2i}\frac{dx^i}{dt} \\
&= eE^2 + e(\vec{v}\times\vec{B})^2 \\
\frac{dp^3}{dt} &= eE^3 + e(\vec{v}\times\vec{B})^3
\end{aligned}$$

confirming the expression

$$\frac{d\vec{p}}{dt} = e[\vec{E} + \vec{v}\times\vec{B}]$$

6.: Suppose person B is moving at constant speed v along the x-axis with respect to person A. Person A throws a ball to person B, and person B throws the ball back immediately. Suppose the speed of the ball (call it u) exceed the speed of light in the thrower's rest frame.

a): Calculate the roundtrip time assuming that person B is a distance L away when he throws the ball back.

answer:

Write the total round trip time as

$$t_{total} = \tau_{out} + \tau_{in}.$$

We are assuming that there is no gravity and friction. By definition, the outgoing time is

$$\tau_{out} = \frac{L}{u}$$

while the return trip time

$$\tau_{in} = \frac{L}{\text{speed in A's frame}} \equiv \frac{L}{u_A}.$$

If the direction of the ball's flight from person A to person B is along the $\hat{x}$ axis, then the return trip ball is moving with velocity $-u_A\hat{x}$:

$$u_A = \frac{-dx}{dt} > 0$$

during return. Now, we need to relate this speed to the speed in the rest frame of B which we know to be u. To accomplish this, we perform the usual Lorentz transformation:

$$\begin{aligned}
dt &= \gamma(d\bar{t} + v\frac{d\bar{x}}{d\bar{t}}d\bar{t}) \\
dx &= \gamma(\frac{d\bar{x}}{d\bar{t}}d\bar{t} + vd\bar{t})
\end{aligned}$$

where the barred frame corresponds to the frame of B. Since are given $\frac{d\bar{x}}{d\bar{t}} = -u$, we see

$$u_A = -\frac{dx}{dt} = \frac{u-v}{1-uv}$$

and consequently

$$\tau_{in} = \frac{L(1-uv)}{u-v}.$$

The roundtrip time is then

$$t_{total} = L\left(\frac{2u - v[1+u^2]}{u[u-v]}\right)$$

b): Can the roundtrip time be negative?

Note that $t_{total} < 0$ if $u > \frac{1+\sqrt{1-v^2}}{v}$. This corresponds to travelling backwards in time.

7): Recall that Minkowski space is a spacetime where the geometry (Lorentz invariant distances) is given by

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$$

where the components of $\eta_{\alpha\beta}$ is given by

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Any time we write a length in the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where dx^μ correspond to an infinitesimal coordinate differential, we call $g_{\mu\nu}$ a metric. Suppose we use the coordinates (u, v, y, z) to cover Minkowski space instead of (t, x, y, z) where $u = t - x$ and $v = t + x$. What are the components of the metric in this space?

answer:

Note that we can write

$$t = \frac{1}{2}[u + v]$$
$$x = \frac{1}{2}[v - u]$$

The initial metric is written as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = -dt^2(u, v) + dx^2(u, v) + dy^2 + dz^2 \\ &= -[du\partial_u t + dv\partial_v t]^2 + [du\partial_u x + dv\partial_v x]^2 + dy^2 + dz^2 \\ &= -\frac{1}{4}[du + dv]^2 + \frac{1}{4}[-du + dv]^2 + dy^2 + dz^2 \\ &= -dudv + dy^2 + dz^2 \end{aligned}$$

Hence, the components of the Minkowski metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the (u, v, y, z) coordinates. Note that the metric is manifestly non-diagonal.