Physics 717 Problem set 3 sol

February 17, 2009

due Monday, Feb 16, 2009, at the beginning of lecture

Problems

1. Using the definition

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta},$$

show that

$$g_{\mu\nu;\lambda} = 0.$$

answer

Using the notation defined in lecture 9, we have

$$g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - \Gamma^\beta_{\lambda\mu} g_{\beta\nu} - \Gamma^\beta_{\lambda\nu} g_{\mu\beta} \tag{1}$$

Using the definition given in the problem

$$\begin{aligned} \partial_\lambda g_{\mu\nu} &= \partial_\lambda \left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu} \right) \eta_{\alpha\beta} \\ &= \left(\partial_\nu \xi^\beta \partial_\lambda \partial_\mu \xi^\alpha + \partial_\mu \xi^\alpha \partial_\lambda \partial_\nu \xi^\beta \right) \eta_{\alpha\beta} \end{aligned}$$

Using lecture 5 result of Weinberg

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \partial_\mu \partial_\nu \xi^\alpha$$

we have

$$\begin{aligned} \partial_\lambda g_{\mu\nu} &= \left(\partial_\nu \xi^\beta \Gamma^\gamma_{\lambda\mu} \partial_\gamma \xi^\alpha + \partial_\mu \xi^\alpha \Gamma^\gamma_{\lambda\nu} \partial_\gamma \xi^\beta \right) \eta_{\alpha\beta} \\ &= g_{\nu\gamma} \Gamma^\gamma_{\lambda\mu} + g_{\mu\gamma} \Gamma^\gamma_{\lambda\nu} \end{aligned}$$

where we have used the definition given in the problem again. Putting this into Eq. (1) have demonstrated $g_{\mu\nu;\lambda} = 0$.

2. Show by explicit construction that two coordinate systems suffice to cover the two-sphere S^2(surface of a ball in Euclidean space).

answer

Consider the stereographic projection of a sphere centered about the origin as shown in Figure 1.

With the ray tracing beginning at the north pole (on positive z-axis), we see that all of the 2-sphere with the exception of the north pole can be mapped to the (x, y) plane. This constitutes one coordinate patch. Similarly, we can start the projection ray tracing from the south pole (on negative z-axis) and project the entire sphere with the exception of the south pole on to the (x, y) plane. This constitutes coordinate patch 2.

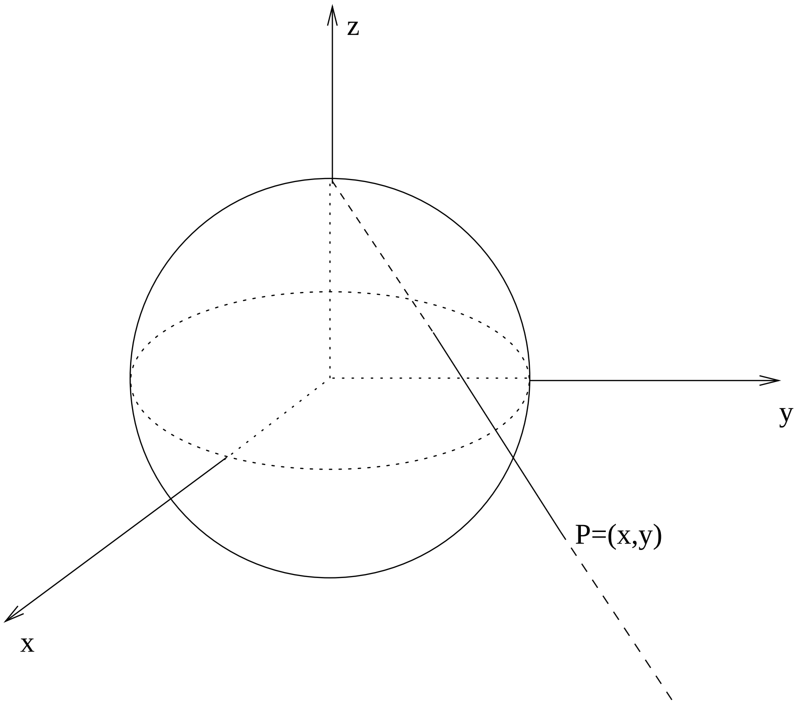

Figure 1: Stereographic projection of 2 sphere centered at the origin.

It is easy to find the explicit coordinate map. Consider the north pole ray patch first. If (X, Y, Z) lie on the sphere (choose to be a unit sphere), then the intersection of the ray with the sphere can be solved by the following equation

$$(0,0,1) + t[(x,y,0) - (0,0,1)] = (X,Y,Z)$$
$$X^2 + Y^2 + Z^2 = 1$$

where t is a parameter for the ray. Explicitly, we have

$$X = tx \tag{2}$$
$$Y = ty \tag{3}$$
$$Z = 1 - t \tag{4}$$
$$t = \frac{2}{1 + x^2 + y^2}. \tag{5}$$

Hence, every point on the sphere (X, Y, Z) except the north pole $(0, 0, 1)$ is mapped to an open patch of the Euclidean space (x_+, y_+) as

$$x_+ = \frac{X}{1-Z}$$
$$y_+ = \frac{Y}{1-Z}.$$

For the south pole patch, we are flipping the Z axis, giving

$$x_- = \frac{X}{1+Z}$$
$$y_- = \frac{Y}{1+Z}.$$

In the overlap region, we have using Eqs. (2), (3), (4), and (5)

$$\psi_+^{-1}(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2})$$

where ψ_+^{-1}denotes the inverse map of chart back onto the points of the manifold (X, Y, Z) on the sphere. Since

$$\psi_-(X,Y,Z) = (\frac{X}{1+Z}, \frac{Y}{1+Z}),$$

the composition

$$\psi_- \circ \psi_+^{-1}(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$$

which is clearly infinitely differentiable except at the origin $(x, y) = (0, 0)$ which is not included in the transition region.

The second patch composition function (transition function) is identical since ψ_+ is identical to ψ_- except for the flip of the sign of Z while $\psi_-^{-1}(x,y)$ flips the sign of the Z component as well (compared to $\psi_+^{-1}(x,y)$).

3. Is

$$\bar{x}(x,t) = x - \frac{1}{2}gt^2$$

$$\bar{t}(x,t) = t$$

an example of a diffeomorphism? Why or why not?

answer

By definition if $f : M \to M'$ is a diffeomorphism if it is a one-to-one and onto map and has a C^∞ inverse. Here, we have $M = (t, x)$ and $M' = (\bar{t}, \bar{x})$ and the charts can be taken to be the defining coordinates themselves. The inverse of the map $f = (t, x - \frac{1}{2}gt^2)$ is given as

$$f^{-1}(\bar{t}, \bar{x}) = (\bar{t}, \bar{x} + \frac{1}{2}g\bar{t}^2).$$

Clearly, this is infinitely differentiable. The map f is also one-to-one and onto since both the f and its inverse are single valued and unique. Hence, f is a diffeomorphism. More mechanically, the matrix

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial \bar{t}}{\partial t} & \frac{\partial \bar{t}}{\partial x} \\ \frac{\partial \bar{x}}{\partial t} & \frac{\partial \bar{x}}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -gt & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -g\bar{t} & 1 \end{pmatrix}$$

has determinant of unity which is well defined everywhere and has the inverse

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \begin{pmatrix} 1 & 0 \\ gt & 1 \end{pmatrix}$$

which is single valued and also infinitely differentiable. Hence, f is a diffeomorphism.

4. Problem 3 b),c) on page 27 of Wald.

answer

b) Expand in basis

$$X = X^\mu \partial_\mu \quad Y = Y^\alpha \partial_\alpha \quad Z = Z^\beta \partial_\beta$$

Now, just write out the Jacobi identity:

$$[X, Y] = X^\mu(\partial_\mu Y^\alpha)\partial_\alpha - Y^\mu(\partial_\mu X^\alpha)\partial_\alpha$$

$$\begin{aligned}[[X,Y],Z] &= X^\mu(\partial_\mu Y^\alpha)(\partial_\alpha Z^\beta)\partial_\beta - Y^\mu(\partial_\mu X^\alpha)(\partial_\alpha Z^\beta)\partial_\beta \\ &\quad -Z^\beta\partial_\beta[X^\mu(\partial_\mu Y^\alpha)\partial_\alpha - Y^\mu(\partial_\mu X^\alpha)\partial_\alpha] \\ &= X^\mu(\partial_\mu Y^\alpha)(\partial_\alpha Z^\beta)\partial_\beta - Y^\mu(\partial_\mu X^\alpha)(\partial_\alpha Z^\beta)\partial_\beta - Z^\beta(\partial_\beta X^\mu)(\partial_\mu Y^\alpha)\partial_\alpha \\ &\quad -Z^\beta X^\mu(\partial_\beta\partial_\mu Y^\alpha)\partial_\alpha + Z^\beta(\partial_\beta Y^\mu)(\partial_\mu X^\alpha)\partial_\alpha + Z^\beta Y^\mu(\partial_\beta\partial_\mu X^\alpha)\partial_\alpha\end{aligned}$$

$$\begin{aligned}[[Y,Z],X] &= Y^\mu(\partial_\mu Z^\alpha)(\partial_\alpha X^\beta)\partial_\beta - Z^\mu(\partial_\mu Y^\alpha)(\partial_\alpha X^\beta)\partial_\beta - X^\beta(\partial_\beta Y^\mu)(\partial_\mu Z^\alpha)\partial_\alpha \\ &\quad -X^\beta Y^\mu(\partial_\beta\partial_\mu Z^\alpha)\partial_\alpha + X^\beta(\partial_\beta Z^\mu)(\partial_\mu Y^\alpha)\partial_\alpha + X^\beta Z^\mu(\partial_\beta\partial_\mu Y^\alpha)\partial_\alpha\end{aligned}$$

$$\begin{aligned}[[Z,X],Y] &= Z^\mu(\partial_\mu X^\alpha)(\partial_\alpha Y^\beta)\partial_\beta - X^\mu(\partial_\mu Z^\alpha)(\partial_\alpha Y^\beta)\partial_\beta - Y^\beta(\partial_\beta Z^\mu)(\partial_\mu X^\alpha)\partial_\alpha \\ &\quad -Y^\beta Z^\mu(\partial_\beta\partial_\mu X^\alpha)\partial_\alpha + Y^\beta(\partial_\beta X^\mu)(\partial_\mu Z^\alpha)\partial_\alpha + Y^\beta X^\mu(\partial_\beta\partial_\mu Z^\alpha)\partial_\alpha\end{aligned}$$

Adding these together, we clearly find all the matching cancellations.

c) The objects $C^\gamma_{\alpha\beta}$ defined by

$$[Y_\alpha, Y_\beta] = C^\gamma_{\alpha\beta} Y_\gamma$$

are called structure constants. Using the Jacobi identity, we write

$$\begin{aligned}[[Y_\alpha,Y_\beta],Y_\delta] + [[Y_\beta,Y_\delta],Y_\alpha] + [[Y_\delta,Y_\alpha],Y_\beta] &= C^\gamma_{\alpha\beta}[Y_\gamma,Y_\delta] + C^\gamma_{\beta\delta}[Y_\gamma,Y_\alpha] + C^\gamma_{\delta\alpha}[Y_\gamma,Y_\beta] \\ &= C^\gamma_{\alpha\beta}C^\lambda_{\gamma\delta}Y_\lambda + C^\gamma_{\beta\delta}C^\lambda_{\gamma\alpha}Y_\lambda + C^\gamma_{\delta\alpha}C^\lambda_{\gamma\beta}Y_\lambda \\ &= 0\end{aligned}$$

Since this holds for any basis vectors Y_λ, we can write

$$C^\gamma_{\alpha\beta}C^\lambda_{\gamma\delta} + C^\gamma_{\beta\delta}C^\lambda_{\gamma\alpha} + C^\gamma_{\delta\alpha}C^\lambda_{\gamma\beta} = 0.$$

5. (Messy problem !)Properties of the affine connection:

a) Compute $\Gamma^\mu_{\alpha\beta}$ for the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

b) What is $g_{\mu\nu}$ for the Minkowski metric if the coordinates are changed from that of part (a) to

$$t' = t$$

$$x' = \sqrt{x^2+y^2}\cos(\phi - wt)$$

$$y' = \sqrt{x^2+y^2}\sin(\phi - wt)$$

$$z' = z$$

where $\tan\phi = y/x$

c) Compute $\Gamma^\mu_{\alpha\beta}$ in the coordinates of part (b).

d) Based on the answers to different parts of this problem, can you tell whether $\Gamma^\mu_{\alpha\beta}$ is a tensor?

answer

a) For the flat Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

since all derivatives vanish, we have $\Gamma^{\mu}_{\alpha\beta} = 0$.

b) The metric in the primed coordinates are

$$g_{\mu'\nu'} = \partial_{\mu'} x^{\alpha} \partial_{\nu'} x^{\beta} g_{\alpha\beta}$$

with

$$\begin{aligned} t' &= t \\ x' &= \sqrt{x^2+y^2}\cos(\phi - wt) \\ y' &= \sqrt{x^2+y^2}\sin(\phi - wt) \\ z' &= z. \end{aligned}$$

Note that the matrix $\partial_{\mu'} x^{\alpha}$ can be written as an inverse through the relation

$$\partial_{\mu'} x^{\alpha} \partial_{\alpha} x^{\nu'} = \delta^{\nu'}{}_{\mu'}$$

Hence, we can write out the matrix

$$T_{\alpha}{}^{\nu} \equiv \partial_{\alpha} x^{\nu'} = \begin{pmatrix} 1 & wr\sin(\phi - wt) & -wr\cos(\phi - wt) & 0 \\ 0 & \frac{x}{r}\cos(\phi - wt) - r\sin(\phi - wt)\partial_x \phi & \frac{x}{r}\sin(\phi - wt) + r\cos(\phi - wt)\partial_x \phi & 0 \\ 0 & \frac{y}{r}\cos(\phi - wt) - r\sin(\phi - wt)\partial_y \phi & \frac{y}{r}\sin(\phi - wt) + r\cos(\phi - wt)\partial_y \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$r \equiv \sqrt{x^2+y^2} = \sqrt{x'^2+y'^2}$$

and the inverse as

$$\partial_{\mu'} x^{\alpha} = (T^{-1})_{\mu'}{}^{\alpha}.$$

Hence, we find

$$g_{\mu'\nu'} = (T^{-1})(g)(T^{-1})^{T} \tag{6}$$

where we have written the right hand side in the usual matrix form. Note that

$$\partial_x \tan\phi = \frac{-y}{x^2} = (\sec^2\phi)\partial_x \phi.$$

Hence,

$$\begin{aligned} \partial_x \phi &= \frac{-y}{x^2}\cos^2\phi \\ &= \frac{-y}{x^2}\frac{1}{1+(\frac{y}{x})^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} \end{aligned}$$

$$\begin{aligned} \partial_y \phi &= \frac{1}{x}\cos^2\phi \\ &= \frac{x}{x^2+y^2} = \frac{x}{r^2} \end{aligned}$$

The inverse matrix T^{-1} can be written as

$$T^{-1} = \begin{pmatrix} 1 & -wy & wx & 0 \\ 0 & \frac{x\cos q + y\sin q}{r} & \frac{y\cos q - x\sin q}{r} & 0 \\ 0 & \frac{x\sin q - y\cos q}{r} & \frac{x\cos q + y\sin q}{r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $q \equiv \phi - wt$. The new metric can be written using Eq. (6) as

$$\begin{aligned} g_{\mu'\nu'} &= \begin{pmatrix} -1+w^2r^2 & -wr\sin q & wr\cos q & 0 \\ -wr\sin q & 1 & 0 & 0 \\ wr\cos q & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1+w^2r^2 & -wy' & wx' & 0 \\ -wy' & 1 & 0 & 0 \\ wx' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where one should note that $r^2 = x'^2 + y'^2$. Note that in the limit that $w \to 0$, we recover the original metric as we should.

An alternative method of solving the problem is to note that

$$r\cos(\phi - wt) = r\cos\phi\cos wt + r\sin\phi\sin wt = x\cos wt + y\sin wt$$

$$r\sin(\phi - wt) = r\sin\phi\cos wt - r\cos\phi\sin wt = y\cos wt - x\sin wt$$

and realize that this is a time dependent coordinate rotation. Hence, one can easily unrotate the system and solve for (x, y) in terms of (x', y'). Of course, the answer is the same.

c) Here, we use the method mentioned at the end of part *b*) because of its relative simplicity. First, write out

$$\begin{aligned} x' &= r\cos(\phi - wt) = r\cos\phi\cos wt + r\sin\phi\sin wt \\ &= x\cos wt + y\sin wt \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\phi - wt) = r\sin\phi\cos wt - r\cos\phi\sin wt \\ &= -x\sin wt + y\cos wt \end{aligned}$$

Solving this for $\{x, y\}$, we find

$$x = x'\cos wt' - y'\sin wt'$$

$$y = x'\sin wt' + y'\cos wt'$$

which as we mentioned before is simply counterclockwise rotation (opposite of clockwise rotation). The remaining variable maps are trivial: $t' = t$ and $z' = z$.

Now, since from part a) we know that $\Gamma^{\lambda}_{\mu\nu} = 0$, we have

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}}$$

$$\Gamma^{0'}_{\mu'\nu'} = \frac{\partial^2 t}{\partial x^{\mu'}\partial x^{\nu'}} = 0$$

$$\Gamma^{3'}_{\mu'\nu'} = \frac{\partial^2 z}{\partial x^{\mu'}\partial x^{\nu'}} = 0$$

$$\begin{aligned} \Gamma^{1'}_{\mu'\nu'} &= \frac{\partial x'}{\partial x^0}\frac{\partial^2 x^0}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial x'}{\partial x}\frac{\partial^2 x}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial x'}{\partial y}\frac{\partial^2 y}{\partial x^{\mu'}\partial x^{\nu'}} \\ &\quad + \frac{\partial x'}{\partial z}\frac{\partial^2 z}{\partial x^{\mu'}\partial x^{\nu'}} \\ &= \cos wt[\delta_{\mu'0}\delta_{\nu'0}(-w^2x) + 2\delta_{(\mu'|0|}\delta_{\nu')1}(-w\sin wt') + 2\delta_{(\mu'|0|}\delta_{\nu')2}(-w\cos wt')] \\ &\quad + \sin wt[\delta_{\mu'0}\delta_{\nu'0}(-w^2y) + 2\delta_{(\mu'|0|}\delta_{\nu')1}(w\cos wt') + 2\delta_{(\mu'|0|}\delta_{\nu')2}(-w\sin wt')] \\ &= -\delta_{\mu'0}\delta_{\nu'0}w^2x' - 2\delta_{(\mu'0}\delta_{\nu')2}w \end{aligned}$$

$$
\begin{aligned}
\Gamma^{2'}_{\mu'\nu'} &= \frac{\partial y'}{\partial x^0}\frac{\partial^2 x^0}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial y'}{\partial x}\frac{\partial^2 x}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial y'}{\partial y}\frac{\partial^2 y}{\partial x^{\mu'}\partial x^{\nu'}} \\
&\quad + \frac{\partial y'}{\partial z}\frac{\partial^2 z}{\partial x^{\mu'}\partial x^{\nu'}} \\
&= -\sin wt'[-w^2 x\delta_{\mu'0}\delta_{\nu'0} + 2\delta_{(\mu'|0|}\delta_{\nu')1}(-w\sin wt') + 2\delta_{(\mu'|0|}\delta_{\nu')2}(-w\cos wt')] \\
&\quad + \cos wt'[\delta_{\mu'0}\delta_{\nu'0}(-w^2 y) + 2\delta_{(\mu'|0|}\delta_{\nu')1}(w\cos wt') + 2\delta_{(\mu'|0|}\delta_{\nu')2}(-w\sin wt')] \\
&= -\delta_{\mu'0}\delta_{\nu'0}w^2 y' + 2w\delta_{(\mu'|0|}\delta_{\nu')1}
\end{aligned}
$$

d) Since if tensor vanishes in one frame, it vanishes in any other frame, $\Gamma^{\mu}_{\alpha\beta}$ is not a tensor since it vanishes in part a) but not in part c).