

Physics 717 Problem set 4

February 23, 2009

due Monday, Feb 23, 2009, at the beginning of lecture

1. Wald Chapter 3, problem 1.

answer

a)

Following lecture 8, all the steps are identical up to bottom of page 5 where

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C_{ab}^c \omega_c$$

where ω_c is a one-form, C_{ab}^c is a (1,2) form, and $\{\tilde{\nabla}, \nabla\}$ are two different derivative operators satisfying properties 1-4 on page 31 of Wald. Now, we take $\tilde{\nabla}$ to be an ordinary covariant derivative and ∇ be a covariant derivative with torsion. Letting

$$\omega_b = \nabla_b f$$

where f is a scalar, we find

$$\nabla_a \nabla_b f = \tilde{\nabla}_a \nabla_b f - C_{ab}^c \nabla_c f$$

$$\nabla_b \nabla_a f = \tilde{\nabla}_b \nabla_a f - C_{ba}^c \nabla_c f.$$

By condition $t(f) = t^a \nabla_a f = t^b \tilde{\nabla}_b f$ (i.e. ∇ and $\tilde{\nabla}$ behaves identically for scalars), we have

$$\nabla_a \nabla_b f = \tilde{\nabla}_a \tilde{\nabla}_b f - C_{ab}^c \nabla_c f$$

$$\nabla_b \nabla_a f = \tilde{\nabla}_b \tilde{\nabla}_a f - C_{ba}^c \nabla_c f.$$

Since by the definition of torsion free, we have $[\tilde{\nabla}_a, \tilde{\nabla}_b]f = 0$, we find

$$[\nabla_a, \nabla_b]f = -2C_{[ab]}^c f.$$

Hence, we have found the torsion tensor

$$T_{ab}^c = C_{ab}^c - C_{ba}^c$$

exists and is well defined.

b)

Start by contracting $X^a Y^b$ to the definition of the torsion tensor:

$$X^a Y^b \nabla_a \nabla_b f - X^a Y^b \nabla_b \nabla_a f = -T_{ab}^c X^a Y^b \nabla_c f$$

Passing the first derivative of the first term through Y^b , we find

$$X^a \nabla_a [Y^b \nabla_b f] - X^a (\nabla_a Y^b) \nabla_b f - X^a Y^b \nabla_b \nabla_a f = -T_{ab}^c X^a Y^b \nabla_c f.$$

The third term can be rewritten as

$$-X^a Y^b \nabla_b \nabla_a f = -Y^b \nabla_b (X^a \nabla_a f) + Y^b (\nabla_b X^a) (\nabla_a f)$$

yielding

$$X^a \nabla_a [Y^b \nabla_b f] - Y^b \nabla_b [X^a \nabla_a f] + Y^b (\nabla_b X^a) (\nabla_a f) - X^a (\nabla_a Y^b) \nabla_b f = -T^c_{ab} X^a Y^b \nabla_c f. \quad (1)$$

Note that since $Y^b \nabla_b f$ is a scalar

$$X^a \nabla_a [Y^b \nabla_b f] = X^a \partial_a (Y^b \partial_b f).$$

Therefore, we can write the first two terms of Eq. (1) as

$$\begin{aligned} X^a \nabla_a [Y^b \nabla_b f] - Y^b \nabla_b [X^a \nabla_a f] &= X^a (\partial_a Y^b) \partial_b f - Y^b (\partial_b X^a) \partial_a f \\ &= [X, Y]^c \partial_c f. \end{aligned}$$

Thus, Eq. (1) becomes

$$X^a (\nabla_a Y^b) \nabla_b f - Y^b (\nabla_b X^a) (\nabla_a f) - [X, Y]^c \nabla_c f = T^c_{ab} X^a Y^b \nabla_c f.$$

Relabeling the indices and dividing out the arbitrary 1-form $\nabla_c f$, we finally arrive at

$$X^a (\nabla_a Y^c) - Y^b (\nabla_b X^c) - [X, Y]^c = T^c_{ab} X^a Y^b.$$

c)

Imposing the condition $\nabla_c g_{ab} = 0$ gives

$$\nabla_c g_{ab} = \partial_c g_{ab} - C^d_{ca} g_{db} - C^d_{cb} g_{ad} = 0.$$

Rewriting this in 3 different ways, we have

$$\begin{aligned} \nabla_c g_{ab} &= \partial_c g_{ab} - C^d_{ca} g_{db} - C^d_{cb} g_{ad} = 0 \\ \nabla_a g_{cb} &= \partial_a g_{cb} - C^d_{ac} g_{db} - C^d_{ab} g_{cd} = 0 \\ \nabla_b g_{ac} &= \partial_b g_{ac} - C^d_{ba} g_{dc} - C^d_{bc} g_{ad} = 0 \end{aligned}$$

Taking a judicious combination as

$$\begin{aligned} \nabla_c g_{ab} + \nabla_a g_{cb} - \nabla_b g_{ac} &= \partial_c g_{ab} - C^d_{ca} g_{db} - C^d_{cb} g_{ad} + \partial_a g_{cb} - C^d_{ac} g_{db} - C^d_{ab} g_{cd} + \\ &\quad - \partial_b g_{ac} + C^d_{ba} g_{dc} + C^d_{bc} g_{ad} \\ &= 0 \end{aligned}$$

Collecting g_{ef} terms, we find

$$\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac} - (C^d_{ac} + C^d_{ca}) g_{db} + (C^d_{ba} - C^d_{ab}) g_{cd} + (C^d_{bc} - C^d_{cb}) g_{ad} = 0. \quad (2)$$

Unlike the torsion-free case, we have the antisymmetric combinations of C not vanishing.

Now, note from the definition of torsion

$$(\partial_a \partial_b f - C^c_{ab} \partial_c f) - (\partial_b \partial_a f - C^c_{ba} \partial_c f) = -T^c_{ab} \partial_c f$$

implying

$$C^c_{ab} - C^c_{ba} = T^c_{ab}.$$

Hence, we can write Eq. (2) as

$$\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac} - (2C^d_{ac} + T^d_{ca}) g_{db} + T^d_{ba} g_{cd} + T^d_{bc} g_{ad} = 0.$$

Solving for C^d_{ac} , we thus find

$$\frac{1}{2} g^{fb} [\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac}] - \frac{1}{2} T^f_{ca} + \frac{1}{2} T^d_{ba} g_{cd} g^{bf} + \frac{1}{2} T^d_{bc} g_{ad} g^{bf} = C^f_{ac}. \quad (3)$$

We can simplify this even further. Since T^c_{ab} is a tensor, we can write

$$C^f_{ac} = \frac{1}{2} g^{fb} [\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac}] + \frac{1}{2} [T^f_{ac} + T^f_{ca} + T^f_{cb}]$$

where we have used the antisymmetry of T^f_{ac} with respect to the lower two indices to switch one of the signs.

2. On the surface of a 2-sphere, the Euclidean metric is given by

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

Suppose a vector $\vec{A} = \hat{e}_\theta$ at $(\theta = \theta_0, \phi = 0)$. What is \vec{A} after it is parallel transported around the circle at $\theta = \theta_0$? What is its magnitude?

answer

The parallel transport equation along the ϕ direction (or \hat{e}_ϕ direction along which $\theta = \theta_0$) can be written as

$$\nabla_2 A^i = \partial_2 A^i + \Gamma_{2k}^i A^k = 0.$$

It is easy to compute the Christoffel symbol:

$$\begin{aligned} \Gamma_{2k}^i &= \frac{1}{2} g^{i\lambda} (g_{2\lambda,k} + g_{k\lambda,2} - g_{2k,\lambda}) \\ &= \frac{-1}{2} \sin(2\theta) \delta_{k2} \delta_{i1} + \cot \theta \delta_{i2} \delta_{k1} \end{aligned}$$

This leads to the differential equation

$$\begin{aligned} \partial_2 A^i &= -\Gamma_{2k}^i A^k \\ &= \left[\frac{1}{2} \delta_{k2} \delta_{i1} \sin(2\theta) - \delta_{i2} \delta_{k1} \cot \theta \right] A^k \end{aligned}$$

or explicitly writing the components, we have

$$\begin{aligned} \partial_2 A^1 &= \frac{1}{2} A^2 \sin(2\theta) \\ \partial_2 A^2 &= -A^1 \cot \theta. \end{aligned} \tag{4}$$

Combining these two equations, we have

$$\partial_2^2 A^2 = -A^2 \cos^2 \theta$$

which has the general solution

$$A^2 = c_1 \sin(\phi \cos \theta) + c_2 \cos(\phi \cos \theta)$$

where c_1 and c_2 are coefficients to be determined by boundary conditions. We are given the boundary conditions

$$A^2(\phi = 0) = 0$$

which implies

$$c_2 = 0$$

and

$$A^2 = c_1 \sin(\phi \cos \theta).$$

Substituting this back into Eq. (4), we find

$$A^1 = -(\tan \theta) \partial_2 A^2 = -c_1 \sin \theta \cos(\phi \cos \theta).$$

Using the boundary condition that $A^1(\phi = 0) = 1 = -c_1 \sin \theta_0$, we find

$$c_1 = \frac{-1}{\sin \theta_0}.$$

Putting these results together, we can write during the parallel transport

$$A^k = \begin{pmatrix} \cos(\phi \cos \theta_0) \\ -\frac{\sin(\phi \cos \theta_0)}{\sin \theta_0} \end{pmatrix}.$$

Setting $\phi = 2\pi$, we arrive at the desired answer:

$$A^k = \begin{pmatrix} \cos(2\pi \cos \theta_0) \\ -\frac{\sin(2\pi \cos \theta_0)}{\sin \theta_0} \end{pmatrix}.$$

The magnitude

$$A^i A^j g_{jk} = \cos^2(2\pi \cos \theta_0) + \sin^2 \theta_0 \frac{\sin^2(2\pi \cos \theta_0)}{\sin^2 \theta_0} = 1$$

remains fixed as expected.

3. (tedious problem) Compute the Riemann tensor component

$$R^0_{101}$$

if the metric is given as

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2).$$

We will see later that this is the average metric for the universe.

answer

Note that we have

$$\begin{aligned} R^a{}_{bcd} &= g^{ae} R_{ebcd} \\ &= g^{ae} R_{cdeb} \\ &= -g^{ae} R_{cdbe} \\ &= R_{dcb}{}^a \end{aligned}$$

Hence, $R^0_{101} = R_{101}{}^0$ and using

$$R_{\gamma\mu\nu}{}^\beta = 2 \left\{ \partial_{[\mu} \Gamma_{\gamma]\nu}{}^\beta + \Gamma_{\nu[\gamma}^\alpha \Gamma_{\mu]\alpha}^\beta \right\}$$

we arrive at

$$R_{101}{}^0 = 2 \left\{ \partial_{[0} \Gamma_{1]1}{}^0 + \Gamma_{1[1}^\alpha \Gamma_{0]\alpha}{}^0 \right\}.$$

Since there is no spatial dependence anywhere, we have

$$R_{101}{}^0 = \partial_0 \Gamma_{11}^0 + \Gamma_{11}^\alpha \Gamma_{0\alpha}^0 - \Gamma_{10}^\alpha \Gamma_{1\alpha}^0$$

We can compute

$$\begin{aligned} \Gamma_{1\alpha}^0 &= \frac{1}{2} g^{0\lambda} (g_{\lambda 1, \alpha} + g_{\lambda \alpha, 1} - g_{1\alpha, \lambda}) \\ &= \frac{1}{2} g^{00} (g_{01, \alpha} + g_{0\alpha, 1} - g_{1\alpha, 0}) \\ &= -\frac{1}{2} g^{00} g_{1\alpha, 0} = \frac{a'}{a} \delta_{\alpha 1} \end{aligned}$$

$$\begin{aligned} \Gamma_{11}^\alpha &= \frac{1}{2} g^{\alpha\lambda} (2g_{\lambda 1, 1} - g_{11, \lambda}) \\ &= -\frac{1}{2} g^{\alpha\lambda} g_{11, \lambda} \\ &= \frac{a'}{a} \delta^{\alpha 0} \end{aligned}$$

$$\Gamma_{0\alpha}^0 = \frac{a'}{a} \delta_{\alpha 0}$$

$$\begin{aligned}\Gamma_{10}^1 &= \frac{1}{2}g^{1\lambda}(g_{\lambda 1,0} + g_{\lambda 0,1} - g_{10,\lambda}) \\ &= \frac{1}{2}g^{11}(g_{11,0}) = \frac{a'}{a}\end{aligned}$$

$$\begin{aligned}R_{101}^0 &= \left(\frac{a'}{a}\right)' + \frac{a'}{a}\frac{a'}{a} - \frac{a'}{a}\frac{a'}{a} \\ &= \left(\frac{a'}{a}\right)'\end{aligned}$$

where the primes are derivatives with respect to η . You could also do this problem using conformal transformation properties discussed in class.

4. Show that

$$R_{abcd} = R_{cdab}$$

answer

From

$$R_{[abc]}^d = 0$$

proved in lecture, we can write

$$R_{abcd} - R_{bacd} - R_{cbad} + R_{bcad} + R_{cabd} - R_{acbd} = 0.$$

Using the antisymmetry property

$$R_{abcd} = -R_{abdc} \tag{5}$$

and

$$R_{abcd} = -R_{bacd}, \tag{6}$$

we can write

$$R_{abcd} + R_{bcad} - R_{acbd} = 0$$

Now, judiciously choose indices (cyclic permutations)

$$R_{abcd} + R_{bcad} - R_{acbd} = 0$$

$$R_{bcd a} + R_{cd b a} - R_{bd c a} = 0$$

$$R_{cd a b} + R_{dac b} - R_{cad b} = 0$$

$$R_{d a b c} + R_{a b d c} - R_{d b a c} = 0$$

Adding the first two lines and subtracting the last two lines give

$$\begin{aligned}R_{abcd} + R_{bcad} - R_{acbd} + R_{bcd a} + R_{cd b a} - R_{bd c a} \\ - R_{cd a b} - R_{dac b} + R_{cad b} - R_{d a b c} - R_{a b d c} + R_{d b a c} = 0\end{aligned}$$

which after trivial cancellations (using antisymmetry properties Eqs. (5) and (6)) become

$$2R_{abcd} + 2R_{cdba} = 0.$$

Hence, we arrive at

$$R_{abcd} = R_{cdab}.$$

5. Consider another common definition of the Riemann tensor $R : V \otimes V \otimes V \rightarrow V$ as

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

where $\nabla_X \equiv X^a \nabla_a$ and $X^a \in V$. Show that this is consistent with the Riemann tensor definition of lecture 9 if there is no torsion.

answer

consider the definition

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

where $\{X, Y, Z\}$ are vector fields. Then in the notation we have been using in class, we find

$$\begin{aligned} R(X, Y, Z) &= (X^a \nabla_a)(Y^b \nabla_b) Z^q - (Y^c \nabla_c)(X^e \nabla_e) Z^q - [X, Y]^c \nabla_c Z^q \\ &= (X^a \nabla_a Y^b)(\nabla_b Z^q) + X^a Y^b \nabla_a \nabla_b Z^q \\ &\quad - (Y^c \nabla_c X^e)(\nabla_e Z^q) - Y^c X^e \nabla_c \nabla_e Z^q \\ &\quad - [X, Y]^c \nabla_c Z^q \\ &= (X^a \nabla_a Y^b)(\nabla_b Z^q) + X^a Y^b [\nabla_a, \nabla_b] Z^q \\ &\quad - (Y^c \nabla_c X^e)(\nabla_e Z^q) - [X, Y]^c \nabla_c Z^q \\ &= T[X, Y]^c \nabla_c Z^q + X^a Y^b [\nabla_a, \nabla_b] Z^q \\ &= T[X, Y]^c \nabla_c Z^q - X^a Y^b R_{ab}{}^q{}_e Z^e \\ &= T[X, Y]^c \nabla_c Z^q + X^a Y^b R_{ab}{}^q{}_e Z^e \end{aligned}$$

Hence, if torsion vanishes (see problem 1), then we recover the definition of lecture 9. $R(X, Y, Z)$ in abstract index notation is $R_{ab}{}^q{}_e$.

6. In Newtonian gravity, two nearby particles with trajectories $x^i(t)$ and $\bar{x}^i(t)$ in Euclidean coordinates evolve as

$$\frac{d^2 \zeta^i}{dt^2} = - \sum_{j=1}^3 \frac{\partial^2 \phi}{\partial x^i \partial x^j} \zeta^j$$

where ϕ is the Newtonian gravitational potential. Use geodesic deviation equation to derive this equation in the weak-field, Newtonian limit of GR, where we have seen that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

can be used with $|h_{\mu\nu}| \ll 1$ and all particle 3-velocity magnitudes can be considered small.

answer

The geodesic deviation equation derived in lecture 10 is

$$a^a = -R_{cb}{}^a{}_d X^b T^c T^d.$$

In a locally inertial frame, $T^c = (1, 0, 0, 0)$ and

$$\begin{aligned} R_{\gamma\mu\nu}{}^\beta &= 2 \left\{ \partial_{[\mu} \Gamma_{\gamma]\nu}^\beta + \Gamma_{\nu[\gamma}^\alpha \Gamma_{\mu]\alpha}^\beta \right\} \\ &\approx 2 \partial_{[\mu} \Gamma_{\gamma]\nu}^\beta \end{aligned}$$

since $\Gamma \sim \mathcal{O}(h)$, making $\Gamma^2 \sim \mathcal{O}(h^2)$ which we neglect. Hence,

$$\begin{aligned} -R_{\gamma\beta\delta}{}^\alpha X^\beta T^\gamma T^\delta &\approx -R_{0\beta 0}{}^\alpha X^\beta \\ &= -2 \partial_{[\beta} \Gamma_{0]0}^\alpha X^\beta \\ &= -\Gamma_{00, \beta}^\alpha X^\beta + \Gamma_{\beta 0, 0}^\alpha X^\beta \\ &= -\Gamma_{00, i}^\alpha X^i \end{aligned}$$

where i sums over 1-3 as usual. The last equality follows from time independence of the Newtonian limit taken in lecture 6. We also found in lecture 6 that

$$\Gamma_{00}^\alpha \approx \frac{-1}{2} \eta^{\alpha i} h_{00, i}.$$

Hence, we arrive at

$$a^j \approx \frac{d^2 X^j}{dt^2} = \frac{1}{2} h_{00,ji} X^i.$$

As we found in lecture 6, the Newtonian identification is $h_{00} \approx -2\phi$ where ϕ is the gravitational potential. Hence, we have arrived at the Newtonian geodesic deviation result:

$$\frac{d^2 X^j}{dt^2} = - \sum_{i=1}^3 \frac{\partial^2 \phi}{\partial x^j \partial x^i} X^i$$