PHYSICS 717 PROBLEM SET 5

due: Monday, March 2, 2009, at the beginning of lecture

Problems

1.: Astronomical observations of the brightness of objects are measurements of the flux of radiation T^{0i} from the object at Earth. Assume there is no gravity (only special relativity applies).

(1) Show that in the rest frame O of a star of constant luminosity L (total energy radiated per second), the stress-energy tensor of the radiation from the star at the event $(t, x, 0, 0)$ has components $T^{00} = T^{0x} = T^{x0} = T^{xx} = \frac{L}{4\pi x^2}$ if the star sits at the origin.

(2) Let X be a null 4-vector which separates the events of emission and reception of the radiation. Show that $X = (x, x, 0, 0)$ in frame O for radiation observed at the event $(x, x, 0, 0)$. Show that the stress-energy tensor of part(a) has the frame-invariant form

$$T = \frac{L}{4\pi} \frac{X \otimes X}{(g_{ab} X^a U^b)^4}$$

where U is the 4-velocity of the star which in the frame O is $(1, 0, 0, 0)$.

(3) Let the Earth-bound observer $\bar{O}$, traveling with speed v away from the star in the x direction, measure the same radiation, again with the star on the $\bar{x}$ axis. Let $X = (R, R, 0, 0)$ in frame $\bar{O}$. Find R as a function of x. Express $T^{\bar{0}\bar{x}}$ in terms of R. In words, physically interpret the results in the limit $v \to 1$.

answer:

a) By definition of

$$T^{\nu\mu} \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} = p^\nu \text{ crossing a surface } dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \tag{0.1}$$

we have

$$T^{00} = \frac{\Delta\text{energy}}{\Delta\text{volume}}$$

which immediately implies

$$T^{00} = \frac{\frac{\Delta\Omega}{4\pi} L \Delta t}{(x^2 \Delta\Omega)\Delta l} = \frac{L \Delta t}{4\pi x^2 \Delta l}$$

where Δl corresponds to radial distance over which the energy spreads in a time Δt and $\Delta\Omega$ is the solid angle subtended at the location $(t, x, 0, 0)$ with respect to the origin. Note that spherical symmetry has been assumed. Since speed of light is $\Delta l / \Delta t = 1$, we have

$$T^{00} = \frac{L}{4\pi x^2}$$

at the spacetime position $(t, x, 0, 0)$.

Similarly, from Eq. (0.1), we have

$$T^{0x} = \frac{\Delta\text{energy}}{(\Delta\text{area transverse to x})(\Delta\text{time})} = \frac{\frac{\Delta\Omega}{4\pi} L \Delta t}{(\Delta\Omega) x^2 \Delta t} = \frac{L}{4\pi x^2}.$$

By symmetry of the stress energy tensor, we have

$$T^{0x} = T^{x0}.$$

Finally, we similarly find

$$T^{xx} = \frac{\Delta p^x}{\Delta\Omega x^2 \Delta t} = \frac{\frac{\Delta\Omega}{4\pi} L \Delta t}{\Delta\Omega x^2 \Delta t} = \frac{L}{4\pi x^2}$$

where we have used the fact that light treated as a massless particle carries momentum in the direction of propagation.

b) As far as the radiation event at $(x, x, 0, 0)$ is concerned, since light travels at speed 1, it must have intersected the spatial origin (emission event) at the event $(0, 0, 0, 0)$. Hence by construction, $(x, x, 0, 0)$ is the null vector X connecting the emission event to the event $(x, x, 0, 0)$.

All we have to do is to evaluate the components of the tensor match the frame invariant expression

$$T = \frac{L}{4\pi} \frac{X \otimes X}{(g_{ab} X^a U^b)^4} \tag{0.2}$$

in one frame (namely frame O in which $U^b = (1, 0, 0, 0)$). At the location $(x, x, 0, 0)$, we find

$$T^{00} = \frac{L}{4\pi} \frac{x^2}{(-x)^4} = \frac{L}{4\pi x^2}$$

which matches the result of part a). Similarly, we find

$$T^{0x} = \frac{L}{4\pi}\frac{x^2}{(-x)^4} = \frac{L}{4\pi x^2} = T^{x0}.$$

Finally, we easily check

$$T^{xx} = \frac{L}{4\pi}\frac{x^2}{(-x)^4} = \frac{L}{4\pi x^2}$$

$$T^{xy} = T^{yx} = 0 = T^{xz} = T^{zx} = T^{yz} = T^{zy}.$$

Hence, the frame invariant form Eq. (0.2) is the correct tensor describing the radiation from the star. (Note that since null events are preserved by Lorentz transformations, this stress tensor is unique for null events.)

c) Lorentz transforming the coordinate $X^\alpha = (t, x, 0, 0)$ in the O frame to $X^{\bar{\alpha}} = (R, R, 0, 0)$ in the frame $\bar{O}$ moving away from the star in the x direction, we have

$$\begin{aligned} R &= \gamma t - v\gamma x \\ R &= \gamma x - v\gamma t \end{aligned}$$

where t can be elimiated (e.g. $t = \frac{R+\gamma v x}{\gamma}$ from the first equation) to yield

$$\begin{aligned} R &= \gamma x - v\gamma(\frac{R+\gamma v x}{\gamma}) \\ &= \gamma x - vR - \gamma v^2 x \end{aligned}$$

or

$$\begin{aligned} R &= x\frac{\sqrt{1-v^2}}{1+v} \\ &= x\frac{\sqrt{1-v}}{\sqrt{1+v}}. \end{aligned}$$

Plugging $X^{\bar{\alpha}} = (R, R, 0, 0)$ into the frame invariant expression for the stress energy tensor, we obtain

$$T^{\bar{0}\bar{x}} = \frac{L}{4\pi}\frac{R^2}{(-RU^{\bar{0}} + RU^{\bar{x}})^4}.$$

Lorentz transformation of $U^\alpha = (1, 0, 0, 0)$ gives

$$U^{\bar{\alpha}} = (\gamma, -\gamma v, 0, 0).$$

Hence, we find

$$\begin{aligned} T^{\bar{0}\bar{x}} &= \frac{L}{4\pi}\frac{R^2}{(-R\gamma - R\gamma v)^4} \\ &= \frac{L}{4\pi R^2}\frac{(1-v^2)^2}{(1+v)^4} \\ &= \frac{L}{4\pi R^2}\frac{(1-v)^2(1+v)^2}{(1+v)^4} \\ &= \frac{L}{4\pi R^2}\frac{(1-v)^2}{(1+v)^2} \end{aligned}$$

which says that as $v \to 1$, we have no energy flux unless $R \to 0$. This is expected because in the limit that $v \to 1$, the light cannot reach the observer unless $R \to 0$.

2.: Show that

$$\nabla_a j^a = 0$$

picks out the Maxwell's equation

$$\nabla^a \nabla_a A_b - R^d{}_b A_d = -j_b$$

instead of

$$\nabla^a \nabla_a A_b = -j_b$$

that one would obtain from the naive substitution rule.

answer:

Start with

$$
\begin{aligned}
\nabla^b[\nabla^a\nabla_a A_b - R^d{}_b A_d] &= \nabla^b\nabla^a\nabla_a A_b - \nabla^b[R^d{}_b A_d] \\
&= [\nabla^b,\nabla^a]\nabla_a A_b + \nabla^a\nabla^b\nabla_a A_b - \nabla^b[R^d{}_b A_d] \\
&= R^{bae}{}_a \nabla_e A_b + R^{bae}{}_b \nabla_a A_e + \\
&\quad \nabla^a\nabla^b\nabla_a A_b - \nabla^b[R^d{}_b A_d] \\
&= R^{be}\nabla_e A_b - R^{ae}\nabla_a A_e + \\
&\quad \nabla^a\nabla^b\nabla_a A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a\nabla^b\nabla_a A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[\nabla^b,\nabla_a]A_b + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[R^b{}_{ab}{}^c A_c] + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[g^{be}g^{fc}R_{eabf}A_c] + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[g^{be}g^{fc}R_{aefb}A_c] + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[g^{fc}R_{af}A_c] + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a[R^c{}_a A_c] + \nabla^a\nabla_a\nabla^b A_b - \nabla^b[R^d{}_b A_d] \\
&= \nabla^a\nabla_a\nabla^b A_b = 0
\end{aligned}
$$

where the last line vanishes because of the Lorentz gauge condition used to derive

$$\nabla^a\nabla_a A_b - R^d{}_b A_d = -j_b.$$

Hence, the term $R^d{}_b A_d$ must be present in the Mawell equations in curved spacetime in Lorentz gauge.

3.: Completing the arguments of lecture 13, show that

$$T^{\alpha\beta}_{(EM)} \equiv F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}$$

satisfies

$$\partial_\alpha T^{\alpha\beta}_{(EM)} = 0$$

if $dF = 0$ and $J^\gamma = 0$ (no external source). Covariantizing this to curved spacetime, is variationally derived Einstein equations consistent with the variational definition of stress energy tensor and

$$S_{EM} = \frac{1}{4}\int d^4x\sqrt{-g}F_{ab}F^{ab}?$$

Why or why not?

answer:

We had from lecture 13 that $d^2F = 0$ implying

$$F_{\gamma\mu,\lambda} + F_{\lambda\gamma,\mu} + F_{\mu\lambda,\gamma} = 0.$$

We also have from that lecture, the intermediate step (when $J^\gamma = 0$)

$$
\begin{aligned}
\partial_\alpha T^\alpha{}_\beta &= F_{\beta\gamma,\alpha}F^{\alpha\gamma} - \frac{1}{2}F^{\gamma\delta}{}_{,\beta}F_{\gamma\delta} \\
&= (-\frac{1}{2}F_{\beta\gamma,\alpha} - \frac{1}{2}F_{\beta\gamma,\alpha} - \frac{1}{2}F_{\gamma\alpha,\beta})F^{\gamma\alpha} \\
&= (-\frac{1}{2}F_{\beta\gamma,\alpha} - \frac{1}{2}F_{\alpha\beta,\gamma} - \frac{1}{2}F_{\gamma\alpha,\beta} + \frac{1}{2}F_{\alpha\beta,\gamma} - \frac{1}{2}F_{\beta\gamma,\alpha})F^{\gamma\alpha} \\
&= (\frac{1}{2}F_{\alpha\beta,\gamma} - \frac{1}{2}F_{\beta\gamma,\alpha})F^{\gamma\alpha} \\
&= (\frac{1}{2}F_{\alpha\beta,\gamma} + \frac{1}{2}F_{\gamma\beta,\alpha})F^{\gamma\alpha} \\
&= 0
\end{aligned}
$$

where the last line follows from the fact that $(\frac{1}{2}F_{\alpha\beta,\gamma} + \frac{1}{2}F_{\gamma\beta,\alpha})$ is symmetric in $\gamma \leftrightarrow \alpha$ while $F^{\gamma\alpha}$ is antisymmetric.

Hence, covariantly (by the $, \rightarrow ;$ rule), the equation of motion for F_{ab} imply

$$\nabla_a T^a{}_b = 0$$

where T_{ab} is the stress energy $F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}$ derived by metric variation. Since $\nabla_a T^a{}_b = 0$ is required by Bianchi identities and Einstein equations, we conclude that variationally derived Einstein equations are consistent with the variational definition of stress energy tensor.

4.: Compute the Einstein tensor component G_{00} for the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\Phi(r)$ and $\Lambda(r)$ are smooth functions of r.

answer:

This is an endurance exercise in algebra. We start with the evaluation of the Christoffel symbols from the metric $ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$:

$$\Gamma^0_{\alpha\beta} = \Phi'(r)(\delta_{\alpha 0}\delta_{\beta r} + \delta_{\beta 0}\delta_{\alpha r})$$

$$\Gamma^r_{\alpha\beta} = \delta_{\alpha r}\delta_{\beta r}2\Lambda'(r) - \frac{1}{2}e^{-2\Lambda}g_{\alpha\beta,r}$$

$$\Gamma^\theta_{\alpha\beta} = \frac{1}{r}(\delta_{\alpha\theta}\delta_{\beta r} + \delta_{\beta\theta}\delta_{\alpha r}) - \sin\theta\cos\theta\delta_{\alpha\phi}\delta_{\beta\phi}$$

$$\Gamma^\phi_{\alpha\beta} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\beta}\delta_{\alpha\phi} + g_{\phi\phi,\alpha}\delta_{\beta\phi})$$

First, work out

$$R_{00} = \Gamma^\mu_{00,\mu} - \Gamma^\mu_{0\mu,0} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{00} - \Gamma^\mu_{0\lambda}\Gamma^\lambda_{\mu 0}.$$

The relevant Christoffel symbols can be expanded as

$$\Gamma^\mu_{00,\mu} = [\frac{-1}{2}e^{-2\Lambda}g_{00,r}]_{,r}$$

$$\Gamma^\lambda_{00} = \delta_{\lambda r}(\frac{-1}{2}e^{-2\Lambda}g_{00,r})$$

$$\Gamma^\mu_{\mu r} = \Phi' + (2\Lambda' - \frac{1}{2}e^{-2\Lambda}g_{rr,r}) + \frac{1}{r} + \frac{1}{2}g^{\phi\phi}g_{\phi\phi,r}$$

$$\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{00} = [\Phi' + \Lambda' + \frac{2}{r}][\Phi' e^{2(\Phi-\Lambda)}]$$

$$\Gamma^\mu_{0\lambda}\Gamma^\lambda_{\mu 0} = 2\Phi'^2 e^{2(\Phi-\Lambda)}$$

Hence, we finally arrive at

$$R_{00} = e^{2(\Phi-\Lambda)}[\Phi'' + \Phi'^2 - \Lambda'\Phi' + \frac{2}{r}\Phi'].$$

The other nonvanishing components can be worked out similarly as

$$R_{rr} = \frac{2\Lambda'}{r} + \Lambda'\Phi' - \Phi'^2 - \Phi''$$

$$R_{\theta\theta} = 1 - e^{-2\Lambda} + e^{-2\Lambda}r\Lambda' - e^{-2\Lambda}r\Phi'$$

$$R_{\phi\phi} = \sin^2\theta + e^{-2\Lambda}(-\sin^2\theta + r\sin^2\theta\Lambda' - r\sin^2\theta\Phi')$$

Combining these, we find

$$R = \frac{2}{r^2} + e^{-2\Lambda}(\frac{-2}{r^2} + \frac{4}{r}\Lambda' - \frac{4}{r}\Phi' + 2\Lambda'\Phi' - 2\Phi'^2 - 2\Phi'')$$

and

$$G_{00} = e^{2\Phi}[\frac{1}{r^2} + e^{-2\Lambda}(\frac{2\Lambda'}{r} - \frac{1}{r^2})]$$

5.: Suppose you are given an action for a complex scalar field Φ charged under electromagnetism as

$$S[g_{\mu\nu}, \Phi, A_\mu] = \int d^4x\sqrt{-g}\left\{-(\partial_\mu\Phi^* + ieA_\mu\Phi^*)(\partial^\mu\Phi - ieA^\mu\Phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2\right\}$$

where $F = dA$ as usual and σ is a number. Compute the total stress energy tensor that couples to gravity. Is the stress tensor symmetric in $\mu \leftrightarrow \nu$? Why or why not?

answer:

We have

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= -g_{\mu\nu}\left\{-(\partial_\alpha\Phi^* + ieA_\alpha\Phi^*)(\partial^\alpha\Phi - ieA^\alpha\Phi) - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2\right\} \\ &\quad -4(\partial\Phi^* + ieA\Phi^*)_{(\mu}(\partial\Phi - ieA\Phi)_{\nu)} + T^{(EM)}_{\mu\nu} \\ &= -4D^*_{(\mu}\Phi^* D_{\nu)}\Phi - g_{\mu\nu}\left\{-(\partial_\alpha\Phi^* + ieA_\alpha\Phi^*)(\partial^\alpha\Phi - ieA^\alpha\Phi) - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2\right\} + T^{(EM)}_{\mu\nu} \end{aligned}$$

where we have symmetrized appropriately since the metric by definition is a symmetric non-degenerate tensor whose variations must also be symmetric (and an extra factor of 2 is from symmetrization convention). By construction, the stress energy tensor is manifestly symmetric in $\mu \leftrightarrow \nu$.

As a further note, let's check the sign of the stress energy tensor. Suppose $A_\mu = 0$ in Minkowski space, and suppose $\partial_\alpha\Phi = \delta_{\alpha 0}\dot{\Phi}$ we have

$$T_{00} = -2|\dot{\Phi}|^2 + |\dot{\Phi}|^2 - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2 = -|\dot{\Phi}|^2 - \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2.$$

Hence, this is a system with energy density $\rho \leq 0$ unbounded from below. Such systems are generically unstable and unphysical. However, if we had defined the action as

$$S[g_{\mu\nu}, \Phi, A_\mu] = \int d^4x\sqrt{-g}\left\{(\partial_\mu\Phi^* + ieA_\mu\Phi^*)(\partial^\mu\Phi - ieA^\mu\Phi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda(|\Phi|^2 - \frac{\sigma^2}{2})^2\right\}$$

the stress tensor would have positive semi-definite energy density.