

PHYSICS 717 PROBLEM SET 6 SOLUTIONS

due: Monday, March 9, 2009, at the beginning of lecture

Problems

1.: Compute the connection 1-forms of the metric

$$ds^2 = -e^{2\Lambda(t,r)} dt^2 + e^{2\Phi(t,r)} dr^2 + e^{2\Gamma(t,r)} (d\theta^2 + \sin^2 \theta d\phi^2).$$

answer:

$$ds^2 = -e^{2\Lambda(t,r)} dt^2 + e^{2\Phi(t,r)} dr^2 + e^{2\Gamma(t,r)} (d\theta^2 + \sin^2 \theta d\phi^2).$$

First define the basis:

$$(e_\mu)_a (e_\nu)_b \eta^{\mu\nu} dx^a dx^b = g_{ab} dx^a dx^b$$

We can then write

$$\begin{aligned} e_0 &= e^\Lambda dt \\ e_1 &= e^\Phi dr \\ e_2 &= e^\Gamma d\theta \\ e_3 &= e^\Gamma \sin \theta d\phi \end{aligned}$$

The torsion free condition equation is then

$$de_\sigma = e_\alpha \wedge w_\sigma{}^\alpha.$$

Write out the components:

$$\begin{aligned} de_0 &= (\partial_r \Lambda) e^\Lambda dr \wedge dt \\ &= e_1 \wedge w_0^1 + e_2 \wedge w_0^2 + e_3 \wedge w_0^3 \\ &= e^\Phi dr \wedge w_0^1 + e^\Gamma d\theta \wedge w_0^2 + e^\Gamma \sin \theta d\phi \wedge w_0^3 \\ de_1 &= (\partial_t \Phi) e^\Phi dt \wedge dr \\ &= e_0 \wedge w_1^0 + e_2 \wedge w_1^2 + e_3 \wedge w_1^3 \\ &= e^\Lambda dt \wedge w_1^0 + e^\Gamma d\theta \wedge w_1^2 + e^\Lambda \sin \theta d\phi \wedge w_1^3 \\ de_2 &= (\partial_t \Gamma) e^\Gamma dt \wedge d\theta + (\partial_r \Gamma) e^\Gamma dr \wedge d\theta \\ &= e_0 \wedge w_2^0 + e_1 \wedge w_2^1 + e_3 \wedge w_2^3 \\ &= e^\Lambda dt \wedge w_2^0 + e^\Phi dr \wedge w_2^1 + e^\Gamma \sin \theta d\phi \wedge w_2^3 \\ de_3 &= (\partial_t \Gamma) e^\Gamma \sin \theta dt \wedge d\phi + (\partial_r \Gamma) e^\Gamma \sin \theta dr \wedge d\phi + e^\Gamma \cos \theta d\theta \wedge d\phi \\ &= e_0 \wedge w_3^0 + e_1 \wedge w_3^1 + e_2 \wedge w_3^2 \\ &= e^\Lambda dt \wedge w_3^0 + e^\Phi dr \wedge w_3^1 + e^\Gamma d\theta \wedge w_3^2 \end{aligned}$$

From the de_3 equation, one can guess

$$\begin{aligned} w_3^2 &= \cot \theta e^{-\Gamma} e_3 \\ w_3^1 &= (\partial_r \Gamma) e^{-\Phi} e_3 \\ w_3^0 &= (\partial_t \Gamma) e^{-\Lambda} e_3 \end{aligned}$$

Using the antisymmetry property of $w_{\mu\nu}$ and raising and lowering indices using $\eta_{\mu\nu}$, we can write for example $w_2^3 = -w_3^2$. Hence, the de_2 equation becomes

$$e^\Lambda dt \wedge w_2^0 + e^\Phi dr \wedge w_2^1 = (\partial_t \Gamma) e^\Gamma dt \wedge d\theta + (\partial_r \Gamma) e^\Gamma dr \wedge d\theta$$

which allows us to guess

$$\begin{aligned} w_2^1 &= e^{-\Phi} (\partial_r \Gamma) e_2 \\ w_2^0 &= e^{-\Lambda} (\partial_t \Gamma) e_2 \end{aligned}$$

Using our guessed solutions for w_3^1 , w_2^1 , w_3^0 , and w_2^0 , we can also simplify the de_1 and de_0 equations as

$$\begin{aligned} e^\Lambda dt \wedge w_1^0 &= (\partial_t \Phi) e^\Phi dt \wedge dr \\ e^\Phi dr \wedge w_0^1 &= (\partial_r \Lambda) e^\Lambda dr \wedge dt \end{aligned}$$

Hence, we must have

$$w_1^0 = w_0^1 = A dr + B dt.$$

Plugging this in and solving for A and B yields

$$A = e^{-\Lambda}(\partial_t \Phi) e^\Phi$$

$$B = e^{-\Phi}(\partial_r \Lambda) e^\Lambda$$

hence

$$w_1^0 = w_0^1 = (\partial_t \Phi) e^{-\Lambda} e_1 + (\partial_r \Lambda) e^{-\Phi} e_0.$$

Hence, we have found all the nonvanishing connection one-forms. All other connection one-forms can be obtained from symmetry property $w_{\mu\nu} = -w_{\nu\mu}$ and raising and lowering indices with $\eta_{\mu\nu}$.

2.: Show that Bianchi identity $\nabla_{[a} R_{bc]d}^e = 0$ follows from

$$dR_\mu{}^\nu = 0.$$

Proof:

Start with

$$R_\mu{}^\nu = d\omega_\mu{}^\nu + \omega_\mu{}^\alpha \wedge \omega_\alpha{}^\nu.$$

Since

$$\begin{aligned} dR_\mu{}^\nu &= d^2\omega_\mu{}^\nu + d\omega_\mu{}^\alpha \wedge \omega_\alpha{}^\nu - \omega_\mu{}^\alpha \wedge d\omega_\alpha{}^\nu \\ &= 0 + (R_\mu{}^\alpha - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\alpha) \wedge \omega_\alpha{}^\nu - \omega_\mu{}^\alpha \wedge (R_\alpha{}^\nu - \omega_\alpha{}^\lambda \wedge \omega_\lambda{}^\nu) \\ &= R_\mu{}^\alpha \wedge \omega_\alpha{}^\nu - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\alpha \wedge \omega_\alpha{}^\nu - \omega_\mu{}^\alpha \wedge R_\alpha{}^\nu + \omega_\mu{}^\alpha \wedge \omega_\alpha{}^\lambda \wedge \omega_\lambda{}^\nu \\ &= R_\mu{}^\alpha \wedge \omega_\alpha{}^\nu - \omega_\mu{}^\alpha \wedge R_\alpha{}^\nu \end{aligned}$$

Putting in basis:

$$\begin{aligned} dR_\mu{}^\nu + \omega_\mu{}^\alpha \wedge R_\alpha{}^\nu - R_\mu{}^\alpha \wedge \omega_\alpha{}^\nu &= 0 \\ \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \omega_\mu{}^\alpha \wedge dx^a \wedge dx^b (R_\alpha{}^\nu)_{ab} - (R_\mu{}^\alpha)_{ef} dx^e \wedge dx^f \wedge \omega_\alpha{}^\nu &= 0 \\ \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \\ (e_\mu)^g \nabla_d (e^\alpha)_g dx^d \wedge dx^a \wedge dx^b (e_\alpha)^e (e^\nu)^f R_{efab} \\ - (e_\mu)^e (e^\alpha)^f R_{efab} dx^a \wedge dx^b \wedge dx^d (e_\alpha)^g \nabla_d (e^\nu)_g &= 0 \end{aligned}$$

Note that

$$\nabla_d[(e^\alpha)_g (e_\alpha)^g] = 0$$

or

$$(e_\alpha)^e \nabla_d (e^\alpha)_g = -(e^\alpha)_g \nabla_d (e_\alpha)^e.$$

Hence, simplifying, we find

$$\begin{aligned} \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \\ (e^\nu)^f R_{efab} dx^d \wedge dx^a \wedge dx^b (e_\mu)^g (e_\alpha)^e \nabla_d (e^\alpha)_g \\ - (e_\mu)^e R_{efab} dx^a \wedge dx^b \wedge dx^d (e_\alpha)^g (e^\alpha)^f \nabla_d (e^\nu)_g &= \\ \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \\ - (e^\nu)^f R_{efab} dx^d \wedge dx^a \wedge dx^b (e_\mu)^g (e_\alpha)^g \nabla_d (e^\alpha)^e \\ - (e_\mu)^e R_{efab} dx^a \wedge dx^b \wedge dx^d g^fg \nabla_d (e^\nu)_g &= \\ \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \\ - (e^\nu)^f R_{efab} dx^d \wedge dx^a \wedge dx^b \eta_{\mu\alpha} \nabla_d (e^\alpha)^e \\ - (e_\mu)^e R_{efab} dx^a \wedge dx^b \wedge dx^d \nabla_d (e^\nu)^f &= \\ \nabla_c((e_\mu)^e (e^\nu)^f R_{efab}) dx^c \wedge dx^a \wedge dx^b + \\ - (e^\nu)^f R_{efab} dx^d \wedge dx^a \wedge dx^b \nabla_d (e_\mu)^e \\ - (e_\mu)^e R_{efab} dx^a \wedge dx^b \wedge dx^d \nabla_d (e^\nu)^f &= \\ (e_\mu)^e (e^\nu)^f (\nabla_c R_{abef}) dx^c \wedge dx^a \wedge dx^b &= 0 \end{aligned}$$

Hence, we have arrived at the desired Bianchi identity.

3.: Express G^0_2 and G^2_2 in terms of Riemann tensor components. E.g. in class we found

$$G^0_0 = -(R^{12}_{12} + R^{13}_{13} + R^{23}_{23}).$$

answer:

As we learned in class, we can read off from the general expression

$$G^\delta_\beta = \frac{-1}{4} R^{\mu\nu}_{\rho\sigma} (3!) \delta^{[\delta}_\beta \delta^\rho_\mu \delta^{\sigma]}_\nu$$

the following components:

$$\begin{aligned} G^0_2 &= \frac{-1}{4} R^{\mu\nu}_{\rho\sigma} (3!) \delta^{[0}_2 \delta^\rho_\mu \delta^{\sigma]}_\nu \\ &= R^{0\nu}_{2\nu} \\ &= R^{01}_{21} + R^{03}_{23} \end{aligned}$$

$$\begin{aligned} G^2_2 &= \frac{-1}{4} R^{\mu\nu}_{\rho\sigma} (3!) \delta^{[2}_2 \delta^\rho_\mu \delta^{\sigma]}_\nu \\ &= -(R^{03}_{03} + R^{13}_{13} + R^{10}_{10}) \end{aligned}$$

4.: Compute the G^0_0 Einstein equation resulting from

$$ds^2 = -dt^2 + a^2(d\chi^2 + \sin^2 \chi [d\theta^2 + \sin^2 \theta d\phi^2]),$$

using the results of lectures 16 and 17.

answer:

Since

$$\begin{aligned} G^0_0 &= -(R^{12}_{12} + R^{13}_{13} + R^{23}_{23}), \\ (R_\mu^\nu)_{cd} &= (e_\mu)_a (e^\nu)_b R^{ab}_{cd}, \end{aligned}$$

and $(e_\mu)_a$ is diagonal in μa , we need to evaluate

$$G^0_0 = -\left(\frac{(R_1^2)_{12}}{(e_1)_1 (e^2)_2} + \frac{(R_1^3)_{13}}{(e_1)_1 (e^3)_3} + \frac{(R_2^3)_{23}}{(e_2)_2 (e^3)_3} \right).$$

we need an expression for ω_3^1 . We need the result from the exercise in lecture 16:

$$\begin{aligned} de_3 &= \dot{a} \sin \chi \sin \theta dt \wedge d\phi + a \cos \chi \sin \theta d\chi \wedge d\phi + a \sin \chi \cos \theta d\theta \wedge d\phi \\ &= \dot{a} \sin \chi \sin \theta e_0 \wedge d\phi + \cos \chi \sin \theta e_1 \wedge d\phi + \cos \theta e_2 \wedge d\phi \\ &= e_0 \wedge \omega_3^0 + e_1 \wedge \omega_3^1 + e_2 \wedge \omega_3^2 \\ \omega_3^0 &= \dot{a} \sin \chi \sin \theta d\phi \\ \omega_3^1 &= \cos \chi \sin \theta d\phi \\ \omega_3^2 &= \cos \theta d\phi \end{aligned}$$

We now simply compute

$$\begin{aligned} R_1^2 &= d\omega_1^2 + \omega_1^\alpha \wedge \omega_\alpha^2 \\ &= d(-\omega_2^1) + \omega_1^0 \wedge \omega_0^2 + \omega_1^3 \wedge \omega_3^2 \\ &= -d(\cos \chi d\theta) + (\dot{a} d\chi) \wedge (\dot{a} \sin \chi d\theta) + (-\cos \chi \sin \theta d\phi) \wedge (\cos \theta d\phi) \\ &= \sin \chi (\dot{a}^2 + 1) d\chi \wedge d\theta \\ R_1^3 &= d\omega_1^3 + \omega_1^\alpha \wedge \omega_\alpha^3 \\ &= d\omega_1^3 + \omega_1^0 \wedge \omega_0^3 + \omega_1^2 \wedge \omega_2^3 \\ &= -d(\cos \chi \sin \theta d\phi) + (\dot{a} d\chi) \wedge (\dot{a} \sin \chi \sin \theta d\phi) + (-\cos \chi d\theta) \wedge (-\cos \theta d\phi) \\ &= \sin \chi \sin \theta d\chi \wedge d\phi - \cos \chi \cos \theta d\theta \wedge d\phi + \dot{a}^2 \sin \chi \sin \theta d\chi \wedge d\phi + \cos \chi \cos \theta d\theta \wedge d\phi \\ &= (1 + \dot{a}^2) \sin \chi \sin \theta d\chi \wedge d\phi \\ R_2^3 &= d\omega_2^3 + \omega_2^0 \wedge \omega_0^3 + \omega_2^1 \wedge \omega_1^3 \\ &= d(-\cos \theta d\phi) + (\dot{a} \sin \chi d\theta) \wedge (\dot{a} \sin \chi \sin \theta d\phi) + (\cos \chi d\theta) \wedge (-\cos \chi \sin \theta d\phi) \\ &= [\sin \theta + \dot{a}^2 \sin^2 \chi \sin \theta - \cos^2 \chi \sin \theta] d\theta \wedge d\phi \end{aligned}$$

where we also used other connection 1-forms derived in lecture 16.

$$\begin{aligned} G^0_0 &= -\left(\frac{\sin \chi(\dot{a}^2 + 1)}{a^2 \sin \chi} + \frac{(1 + \dot{a}^2) \sin \chi \sin \theta}{aa \sin \chi \sin \theta} + \frac{\sin \theta + \dot{a}^2 \sin^2 \chi \sin \theta - \cos^2 \chi \sin \theta}{a \sin \chi a \sin \chi \sin \theta}\right) \\ &= -\left(\frac{\dot{a}^2 + 1}{a^2} + \frac{(1 + \dot{a}^2)}{a^2} + \frac{1 + \dot{a}^2}{a^2}\right) \\ &= -3 \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} \right] = 8\pi T^0_0 \end{aligned}$$

Alternatively, one can express both sides of Einstein equations in the tetrad basis.

5.: Consider a 2-sphere of unit radius with the natural metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

Find 3 linearly independent Killing vectors.

answer:

$$\begin{aligned} \xi_{\alpha;\beta} + \xi_{\beta;\alpha} &= 0 \\ \xi_{\alpha,\beta} + \xi_{\beta,\alpha} - 2\Gamma_{\alpha\beta}^\mu \xi_\mu &= 0 \\ \Gamma_{\alpha\beta}^1 &= \frac{1}{2} g^{1\lambda} (g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \\ &= -\delta_{\alpha 2} \delta_{\beta 2} \sin \theta \cos \theta \\ \Gamma_{\alpha\beta}^2 &= \frac{1}{2} g^{2\lambda} (g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \\ &= (\delta_{\alpha 2} \delta_{\beta 1} + \delta_{\beta 2} \delta_{\alpha 1}) \cot \theta \end{aligned}$$

Hence, the Killing equations are

$$\xi_{1,2} + \xi_{2,1} - 2\Gamma_{12}^2 \xi_2 = \xi_{1,2} + \xi_{2,1} - 2 \cot \theta \xi_2 = 0$$

$$(0.1) \quad \xi_{1,1} = 0$$

$$(0.2) \quad \xi_{2,2} + \sin \theta \cos \theta \xi_1 = 0$$

Eq. (0.1) gives

$$\xi_1 = \xi_1(\phi)$$

while Eq. (0.2) gives

$$\xi_{2,2}(\theta, \phi) = -\xi_1(\phi) \frac{1}{2} \sin 2\theta.$$

Hence,

$$(0.3) \quad \xi_2(\theta, \phi) = -\frac{1}{2} \sin 2\theta \int^\phi \xi_1(\phi') d\phi' + g(\theta)$$

Inserting this into the remaining equation gives

$$\xi_{1,2} + \xi_{2,1} - 2 \cot \theta \xi_2 = 0$$

$$(0.4) \quad \xi_1'(\phi) + \int^\phi \xi_1(\phi') d\phi' + g'(\theta) - 2 \cot \theta g(\theta) = 0$$

Differentiating this again with respect to ϕ gives

$$\xi_1''(\phi) + \xi_1(\phi) = 0.$$

Solving for $\xi_1(\phi)$ gives

$$\xi_1(\phi) = a \cos \phi + b \sin \phi.$$

Eq. (0.4) also gives

$$g'(\theta) - 2 \cot \theta g(\theta) = 0$$

which has a simple solution

$$g(\theta) = c \sin^2 \theta.$$

Hence, we have the remaining solution given by Eq. (0.3) as

$$\xi_2(\theta, \phi) = -\frac{1}{2} \sin 2\theta [a \sin \phi - b \cos \phi] + c \sin^2 \theta.$$

Since we have 3 adjustable constants, we have 3 independent Killing vectors. Explicitly, with $a = b = 0$, we have

$$\xi^\mu = (0, c) = c(\partial_\phi)^\mu.$$

With $c = 0 = b$, we have

$$\xi^\mu = a(\cos \phi, -\cot \theta \sin \phi).$$

Finally, with $c = 0 = a$, we have

$$\xi^\mu = b(\sin \phi, \cot \theta \cos \phi).$$

These are all proportional to generators of the angular momentum.

6.: Show that any Killing vector ξ^ν is a solution of the equation

$$\nabla^\mu \nabla_\mu \xi^\nu + R^\nu{}_\sigma \xi^\sigma = 0.$$

Compare and contrast with the equation for the vector potential of electrodynamics.

answer:

Start with the definition

$$\nabla_{(\mu} \xi_{\nu)} = 0.$$

This gives

$$\begin{aligned} \nabla^\mu (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) &= \nabla^\mu \nabla_\mu \xi_\nu + \nabla^\mu \nabla_\nu \xi_\mu \\ &= \nabla^\mu \nabla_\mu \xi_\nu + g^{\mu\lambda} [\nabla_\lambda, \nabla_\nu] \xi_\mu + g^{\mu\lambda} \nabla_\nu \nabla_\lambda \xi_\mu \\ &= \nabla^\mu \nabla_\mu \xi_\nu + g^{\mu\lambda} R_{\lambda\nu\mu}{}^\beta \xi_\beta + g^{\mu\lambda} \nabla_\nu \nabla_\lambda \xi_\mu \\ &= \nabla^\mu \nabla_\mu \xi_\nu + R_\nu{}^\beta \xi_\beta + \nabla_\nu \nabla^\mu \xi_\mu \\ &= 0 \end{aligned}$$

Since

$$g^{\mu\nu} \nabla_{(\mu} \xi_{\nu)} = 0 = \nabla^\mu \xi_\mu$$

we have found

$$\nabla^\mu \nabla_\mu \xi^\nu + R^\nu{}_\beta \xi^\beta = 0.$$

This equation is very similar to the vacuum equation of electrodynamics in Lorentz gauge when we identify $\xi^\nu \rightarrow A^\nu$, except in that case, the equation Ricci tensor has the opposite sign:

$$\nabla^\mu \nabla_\mu A^\nu - R^\nu{}_\beta A^\beta = 0.$$