Lec 17 : Einstein Eq. in tetrad formalism

Note Title 3/1/2009

Aim : Express Einstein eq. in tetrad formalism.

Recall the formula used before :

$$\epsilon^{a_1\ldots a_j a_{j+1}\ldots a_n}\epsilon_{a_1\ldots a_j b_{j+1}\ldots b_n} = (-1)(n-j)!j!\delta^{[a_{j+1}}_{b_{j+1}}\ldots\delta^{a_n]}_{b_n}$$

Use it to simplify the following :

$$\begin{aligned}
\frac{1}{(2!)^2}\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\rho\sigma}\epsilon^{\rho\sigma\alpha\delta} &= \frac{1}{4}R^{\mu\nu}{}_{\rho\sigma}\epsilon^{\alpha\delta\rho\sigma}\epsilon_{\alpha\beta\mu\nu} \\
&= -\frac{1}{4}R^{\mu\nu}{}_{\rho\sigma}(3!)\delta^{[\delta}{}_{\beta}\delta^{\rho}{}_{\mu}\delta^{\sigma]}{}_{\nu} \\
&= -\frac{1}{4}(3!)R^{[\rho\sigma}{}_{\rho\sigma}\delta^{\delta]}{}_{\beta} \\
&= -\frac{1}{4}\{R^{\rho\sigma}{}_{\rho\sigma}\delta^{\delta}{}_{\beta} - R^{\sigma\rho}{}_{\rho\sigma}\delta^{\delta}{}_{\beta} - R^{\delta\sigma}{}_{\rho\sigma}\delta^{\rho}{}_{\beta} + R^{\sigma\delta}{}_{\rho\sigma}\delta^{\rho}{}_{\beta} + R^{\delta\rho}{}_{\rho\sigma}\delta^{\sigma}{}_{\beta} - R^{\rho\delta}{}_{\rho\sigma}\delta^{\sigma}{}_{\beta}\} \\
&= -\frac{1}{4}\{2R\delta^{\delta}{}_{\beta} - R^{\delta\sigma}{}_{\beta\sigma} + R^{\sigma\delta}{}_{\beta\sigma} + R^{\delta\rho}{}_{\rho\beta} - R^{\rho\delta}{}_{\rho\beta}\} \\
&= -\frac{1}{4}\{2R\delta^{\delta}{}_{\beta} - 4R^{\delta}{}_{\beta}\} \\
&= G^{\delta}{}_{\beta} \quad \leftarrow \text{Einstein tensor!}
\end{aligned}$$

Next, we use an intermediate result!

$$G^{\delta}{}_{\beta} = -\frac{1}{4} R^{\mu\nu}{}_{\rho\sigma} (3!) \delta^{[\delta}_{\beta} \delta^{\rho}_{\mu} \delta^{\sigma]}_{\nu}$$

$$\begin{aligned} G^{0}{}_{0} &= -\frac{1}{4} R^{\mu\nu}{}_{\rho\sigma} (3!) \delta^{[0}_{0} \delta^{\rho}_{\mu} \delta^{\sigma]}_{\nu} \\ &= -(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}) \end{aligned}$$

<u>Proof</u>

$$a\,b\,c - a\,c\,b - c\,b\,a$$
$$- b\,a\,c + b\,c\,a + c\,a\,b$$

Consider the part of the sum w/ $\rho = 0$:

$$R^{\mu\nu}{}_{0\sigma}\, \delta^{[0}_{0}\, \delta^{0}_{\mu}\, \delta^{\sigma]}_{\nu}$$

$$= \begin{smallmatrix}0&0&\sigma\\0&\mu&\nu\end{smallmatrix} - \begin{smallmatrix}0&\sigma&0\\0&\mu&\nu\end{smallmatrix} - \begin{smallmatrix}\sigma&0&0\\0&\mu&\nu\end{smallmatrix}$$
$$- \begin{smallmatrix}0&0&\sigma\\0&\mu&\nu\end{smallmatrix} + \begin{smallmatrix}0&\sigma&0\\0&\mu&\nu\end{smallmatrix} + \begin{smallmatrix}\sigma&0&0\\0&\mu&\nu\end{smallmatrix}$$

switched

$$= 0$$

Similarly, if $\sigma = 0$, $0 \leftrightarrow \sigma$ switch + antisymm cancels.

More generally, no index among $\{0, \rho, \sigma\}$ in $\delta^{[0}_{0}\delta^{\rho}{}_{\mu}\delta^{\sigma]}_{\nu}$ can be the same.

$$R^{\mu\nu}{}_{\rho\sigma}(3!)\delta^{[0}_{0}\delta^{\rho}{}_{\mu}\delta^{\sigma]}_{\nu} = \sum_{\rho\neq\sigma=1}^{3} R^{\mu\nu}{}_{\rho\sigma}\,\delta^{\rho}{}_{\mu}\,\delta^{\sigma}{}_{\nu} - \sum_{\rho\neq\sigma=1}^{3} R^{\mu\nu}{}_{\rho\sigma}\,\delta^{\sigma}{}_{\mu}\,\delta^{\rho}{}_{\nu}$$

$$\sum_{\rho\neq\sigma=1}^{3} R^{\mu\nu}{}_{\rho\sigma}\delta^{\rho}{}_{\mu}\delta^{\sigma}{}_{\nu} = R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23} + R^{21}{}_{21} + R^{31}{}_{31} + R^{32}{}_{32}$$
$$= 2\left(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}\right)$$

Similarly

$$-\sum_{\rho\neq\sigma=1}^{3} R^{\mu\nu}{}_{\rho\sigma}\,\delta^{\sigma}{}_{\mu}\,\delta^{\rho}{}_{\nu} = -\left(R^{21}{}_{12} + R^{31}{}_{13} + R^{32}{}_{23} + R^{12}{}_{21} + R^{13}{}_{31} + R^{23}{}_{32}\right)$$
$$= 2\left(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}\right)$$

$$\therefore\ R^{\mu\nu}{}_{\rho\sigma}(3!)\delta^{[0}_{0}\delta^{\rho}{}_{\mu}\delta^{\sigma]}_{\nu} = 4\left(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}\right)$$

$$\therefore\ \boxed{G^{0}{}_{0} = -\left(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}\right)}$$

Q: Find an expression for $G^1{}_1$.

$$G^1{}_2 = -\frac{1}{4} R^{\mu\nu}{}_{\rho\sigma}(3!)\,\delta^{[1}{}_2\,\delta^\rho{}_\mu\,\delta^{\sigma]}{}_\nu$$

$$= -(R^{01}{}_{20} + R^{31}{}_{23})$$

proof

$$R^{\mu\nu}{}_{\rho\sigma}(3!)\,\delta^{[1}{}_2\,\delta^\rho{}_\mu\,\delta^{\sigma]}{}_\nu = R^{\mu\nu}{}_{\rho\sigma}(3!)\,\delta^{[\rho}{}_2\,\delta^\sigma{}_\mu\,\delta^{1]}{}_\nu$$

$$= R^{\mu\nu}{}_{\rho\sigma}\,\delta^\rho{}_2\,\delta^\sigma{}_\mu\,\delta^1{}_\nu - R^{\mu\nu}{}_{\rho\sigma}\,\delta^\sigma{}_2\,\delta^\rho{}_\mu\,\delta^1{}_\nu$$

$$- R^{\mu\nu}{}_{\rho\sigma}\,\delta^\rho{}_2\,\delta^1{}_\mu\,\delta^\sigma{}_\nu - R^{\mu\nu}{}_{\rho\sigma}\,\underbrace{\cancel{\delta^1{}_2}}_{0}\,\delta^\sigma{}_\mu\,\delta^\rho{}_\nu$$

$$+ R^{\mu\nu}{}_{\rho\sigma}\,\delta^\sigma{}_2\,\delta^1{}_\mu\,\delta^\rho{}_\nu + R^{\mu\nu}{}_{\rho\sigma}\,\underbrace{\cancel{\delta^1{}_2}}_{0}\,\delta^\rho{}_\mu\,\delta^\sigma{}_\nu$$

$$= R^{\mu 1}{}_{2\mu} - R^{\rho 1}{}_{\rho 2} - R^{1\sigma}{}_{2\sigma} + R^{1\rho}{}_{\rho 2}$$

$$= 2\left(R^{\mu 1}{}_{2\mu} + R^{1\rho}{}_{\rho 2}\right) = 2\left[\left(R^{01}{}_{20} + R^{31}{}_{23}\right) + \left(R^{10}{}_{02} + R^{13}{}_{32}\right)\right]$$

$$= 4\left[R^{01}{}_{20} + R^{31}{}_{23}\right]$$

$$\therefore \quad \boxed{G^1{}_2 = -\left[R^{01}{}_{20} + R^{31}{}_{23}\right]}$$

$$
\begin{aligned}
G^0{}_3 &= -\frac{1}{4} R^{\mu\nu}{}_{\rho\sigma}(3!)\delta^{[0}_3 \delta^{\rho}{}_{\mu} \delta^{\sigma]}_{\nu} \\
&= -\frac{1}{4} R^{\mu\nu}{}_{\rho\sigma}(3!)\delta^{[\rho}_3 \delta^{\sigma}{}_{\mu} \delta^{0]}_{\nu} \\
&= -(R^{10}{}_{31} + R^{20}{}_{32})
\end{aligned}
$$

bottom cannot contain 0
top cannot cotain 3

All other components are straight forward generalizations.

Example

$$ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$G^0{}_0 = -(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23})$$

$$= -\left(\frac{h'}{2rh^2} + \frac{h'}{2rh^2} + \frac{-1}{r^2}\frac{1}{h} + \frac{1}{r^2}\right) = 0$$

$$\frac{h'}{rh^2} - \frac{1}{r^2}\frac{1}{h} + \frac{1}{r^2} = 0$$

$$h' - \frac{h}{r} + \frac{h^2}{r} = 0$$

$$\frac{dh}{dr} = \frac{h - h^2}{r}$$

$$\int \frac{dh}{h-h^2} = \ln r + c$$

$$\int dh \left[\frac{a}{h} + \frac{b}{1-h} \right] = \ln r + c$$

$$a - ah + bh = 1 \quad \Rightarrow \quad a = b \qquad a = 1$$

$$\ln h - \ln(1-h) = \ln r + c$$

$$\frac{h}{1-h} = e^{c} r \equiv \tilde{c}\, r$$

$$h = \tilde{c}\, r\,(1-h)$$

$$h\left[1 + \tilde{c}\, r\right] = \tilde{c}\, r$$

$$h = \frac{\tilde{c}\, r}{1 + \tilde{c}\, r} = \boxed{\frac{1}{1 + \frac{1}{\tilde{c}\, r}}}$$

Later, you will see that $\tilde{c} = \frac{-1}{2M}$ to give Schwarzschild metric:

Some other minor technique

$$\Gamma^{\alpha}{}_{\alpha\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\beta,\mu} + g_{\mu\beta,\alpha} - g_{\alpha\mu,\beta}\right)$$

$$= \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\,\delta g_{\alpha\beta}$$

$$\partial_\mu \ln\left(\sqrt{-g}\right) = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \cancel{\frac{1}{\sqrt{-g}}}\,\frac{1}{2}\cancel{\sqrt{-g}}\, g^{\alpha\beta}\,\partial_\mu g_{\alpha\beta}$$

$$= \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu}$$

$$= \boxed{\Gamma^{\alpha}{}_{\alpha\mu}}$$