Lec 40: Stress energy tensor by variation of tetrad

Note Title 4/30/2009

Last lecture:

$$\boxed{D_\nu = (e_\nu)^a \left[\partial_a + \Gamma_a \right] \qquad \Gamma_a = \tfrac{1}{2} \sigma^{\alpha\beta} (e_\alpha)^b \nabla_b (e_\beta)_a}$$

with $SL(2,\mathbb{C})$ indices displayed

$$(D_\nu)_\alpha{}^\beta = (e_\nu)^a \left[\delta_\alpha^\beta \partial_a + (\Gamma_a)_\alpha{}^\beta \right]$$

Any Lorentz group rep.

$D_\nu \psi$ transforms as a $D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},0)}$ tensor

Q: Why?

Q: Suppose $\sigma^{\alpha\beta}$ is given by $\Sigma^{\alpha\beta} \equiv \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$ and ψ has the corresponding rep. Then what is the transformation property of $D_\nu \psi$?

Q: What is the action in that case?

An application: Derivation of stress tensor of spin $\frac{1}{2}$ field.

$$\frac{\delta S}{\delta (e_\alpha)^c} = \frac{\partial g^{ab}}{\partial (e_\alpha)^c}\,\frac{\delta S}{\delta g^{ab}} = \frac{\partial g^{ab}}{\partial (e_\alpha)^c}\,\frac{\sqrt{g}}{2}\,T_{ab}$$

$$g^{ab} = (e^\alpha)^a (e_\alpha)^b$$

$$d\,g^{ab} = d(e^\alpha)^a (e_\alpha)^b + (e^\alpha)^a d(e_\alpha)^b$$

$$= d(e_\alpha)^c\,\delta_c{}^a (e^\alpha)^b + (e^\alpha)^a d(e_\alpha)^c\,\delta_c{}^b$$

$$\frac{\partial g^{ab}}{\partial (e_\alpha)^c} = \delta_c{}^a (e^\alpha)^b + (e^\alpha)^a \delta_c{}^b$$

$$\frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta (e_\alpha)^c} = \left[\delta_c{}^a (e^\alpha)^b + (e^\alpha)^a \delta_c{}^b\right] T_{ab}$$

$$= T_{cb} (e^\alpha)^b + (e^\alpha)^a T_{ac}$$

$$\frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta (e_\alpha)^c}(e_\alpha)_f = T_{cb} (e^\alpha)^b (e_\alpha)_f + (e^\alpha)^a (e_\alpha)_f\,T_{ac}$$

$$= T_{cf} + T_{fc} = 2\,T_{cf}$$

Note $\sqrt{g} = \sqrt{|\det[(e^\alpha)_a (e_\alpha)_b]|}$

If $M^\alpha{}_a \equiv e^\alpha{}_a$, $\sqrt{g} = \sqrt{|\det(M \eta M)|} = \sqrt{(\det M)^2} = |\det M|$

$\therefore \quad \sqrt{g} = |\det(e^\alpha{}_a)| \equiv e$

$$\boxed{S = \int d^4x\; e\; \mathcal{L}}$$

$$\therefore \quad \boxed{T_{cf} = \frac{1}{e}\,\frac{\delta S}{\delta (e_\alpha)^c}(e_\alpha)_f}$$

How do we evaluate δe ?

Using $\delta(\det A) = (\det A)\,\mathrm{Tr}(A^{-1}\,\delta A)$

and noting $(e_\alpha)^b (e^\alpha)_a = \delta^b{}_a$

$$\delta e = e\,(e_\alpha)^a\,\delta(e^\alpha)_a$$

Since $(e_\alpha)^a (e^\alpha)_b = \delta^a{}_b$

$$\delta(e_\alpha)^a (e^\alpha)_b + (e_\alpha)^a\,\delta(e^\alpha)_b = 0 \Rightarrow \delta(e^\lambda)_b = -(e^\lambda)_a\,\delta(e_\alpha)^a (e^\alpha)_b$$

$$\therefore \quad \delta e = -e\,(e_\alpha)^a (e^\alpha)_c\,\delta(e_\gamma)^c (e^\gamma)_a$$

$$= \boxed{-e\,(e^\gamma)_c\,\delta(e_\gamma)^c}$$

Example

$$S = \int d^4x\,\sqrt{g}\,\tfrac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\,g^{\mu\nu} \qquad (+1, -1, -1, -1)$$

$$\frac{\delta S}{\delta g^{\alpha\beta}} = \frac{\sqrt{g}}{2}\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}\sqrt{g}\,g_{\alpha\beta}\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\right)$$

$$T_{\alpha\beta} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}} = \partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(\partial_\mu\phi\,\partial^\mu\phi)$$

Using tetrad, we have

$$S = \int d^4x\; e\;\tfrac{1}{2}(\partial_a\phi)(\partial_b\phi)(e_\alpha)^a(e^\alpha)^b$$

$$\frac{\delta S}{\delta(e_\alpha)^c} = e\,(\partial_c\phi)(\partial_b\phi)(e^\alpha)^b - e\,(e^\alpha)_c\left[\tfrac{1}{2}(\partial_a\phi)(\partial^a\phi)\right]$$

$$\therefore \quad T_{cf} = \frac{1}{e}\left\{e\,(\partial_c\phi)(\partial_b\phi)(e^\alpha)^b - e\,(e^\alpha)_c\left[\tfrac{1}{2}(\partial_a\phi)(\partial^a\phi)\right]\right\}(e_\alpha)_f$$

$$= (\partial_c\phi)(\partial_f\phi) - g_{cf}\left[\tfrac{1}{2}(\partial_a\phi)(\partial^a\phi)\right] \quad \checkmark$$

example

$$S = \int d^4x \; e \; \overbrace{i \bar{\psi}_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\alpha} (e_\mu)^a \left[\delta_\alpha{}^\beta \partial_a + \tfrac{1}{2} (\sigma^{\delta\epsilon})_\alpha{}^\beta (e_\delta)^b \nabla_b (e_\epsilon)_a \right] \psi_\beta}^{\mathcal{L}}$$

$$\frac{\delta S}{\delta (e_\gamma)^c} = e \, i \bar{\psi}_{\dot\alpha} (\bar\sigma^{\gamma})^{\dot\alpha\alpha} \left[\delta_\alpha{}^\beta \partial_c + \tfrac{1}{2} (\sigma^{\delta\epsilon})_\alpha{}^\beta (e_\delta)^b \nabla_b (e_\epsilon)_c \right] \psi_\beta$$

$$+ \, e \, i \bar{\psi}_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\alpha} (e_\mu)^a \left[\tfrac{1}{2} (\sigma^{\gamma\epsilon})_\alpha{}^\beta \nabla_c (e_\epsilon)_a \right] \psi_\beta$$

integration by parts

$$+ \, - i e \nabla_b \left\{ \bar{\psi}_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\alpha} (e_\mu)_c \left[\tfrac{1}{2} (\sigma^{\delta\gamma})_\alpha{}^\beta (e_\delta)^b \right] \psi_\beta \right\} - e (e^\gamma)_c \mathcal{L}$$

$$\boxed{\begin{aligned} T_{cf} = (e_\gamma)_f \Big\{ & i \bar{\psi}_{\dot\alpha} (\bar\sigma^{\gamma})^{\dot\alpha\alpha} (D_c)_\alpha{}^\beta \psi_\beta + i \bar{\psi}_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\alpha} (e_\mu)^a \left[\tfrac{1}{2} (\sigma^{\gamma\epsilon})_\alpha{}^\beta \nabla_c (e_\epsilon)_a \right] \psi_\beta \\ & - i \nabla_b \left\{ \bar{\psi}_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\alpha} (e_\mu)_c \left[\tfrac{1}{2} (\sigma^{\delta\gamma})_\alpha{}^\beta (e_\delta)^b \right] \psi_\beta \right\} \Big\} - g_{fc} \mathcal{L} \end{aligned}}$$